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Defect 3 Blocks of Symmetric Group Algebras

Stuart Martin

Magdalene College, Cambridge, CB3 0AG, England

and

Lee Russell*

Pembroke College, Cambridge, CB2 1RF, England

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INTRODUCTION

Let k be a field of characteristic $p \geq 5$ and let $\mathfrak{S}_n$ denote the symmetric group of degree $n \in \mathbb{N}$. In [7], we carried out a detailed study of the principal block of $k\mathfrak{S}_{3p}$, which is the unique block of defect 3 for that group algebra. In particular, we found that all the decomposition numbers of this block are 0 or 1, that there are no self-extensions of irreducibles, and that all extensions between irreducibles are 0 or one-dimensional. The aim of this paper is to show that these properties hold for all defect 3 blocks of symmetric group algebras. We also find that the Ext-quivers of defect 3 blocks are all similar in shape. Thus, as with the principal block in the defect 2 case (see Martin [6]), the principal block of $k\mathfrak{S}_{3p}$ will prove to be a prototype for all blocks of defect 3.

We start by extending Scopes' analysis of [2:1] pairs to blocks of defect 3 (namely [3:1] and [3:2] pairs) and show how part of the decomposition matrix and quiver of certain defect 3 blocks of $k\mathfrak{S}_n$ can be determined from blocks of $k\mathfrak{S}_{n-1}$ and $k\mathfrak{S}_{n-2}$. It then follows immediately that the properties mentioned above hold for nearly all cases of a defect 3 block B of some symmetric group algebra $k\mathfrak{S}_n$, if it is assumed that these proper-

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ties hold for all defect 3 blocks of algebras of smaller symmetric groups. The exceptional cases are those blocks whose p-core is rectangular, that is, of the form (x^z), or made up of two rectangles, that is, of the form $(x_1^{z_1}, x_2^{z_2})$. In the latter case we shall call this core "birectangular." These cases are considered separately. The main theorem is stated in Section 7. Further work on such pairs will appear in [13].

1. BACKGROUND

Facts on modular representations of finite groups can be found in Benson [1] and Landrock [5]. A general survey of the representation theory of the symmetric group is given in James and Kerber [3]. Our notation and conventions are drawn from there. In particular, we label partitions by using the device of James' abacus [3, pp. 77–79]. This is also summarized in [7] together with a statement of Schaper's formula (first proved in [10] and reformulated in [4]).

Suppose λ is a partition whose Specht module belongs to a block of defect 3 and whose Young diagram has less than or equal to m rows. Suppose also that when represented on an abacus with p runners using m beads, there are m_j beads on the jth runner for each j. The $\langle m_1, \ldots, m_p \rangle$ notation is then defined as follows.

DEFINITION 1.1. (1) If the m-bead abacus for λ has a single bead of weight 1 on different runners i, j, and l, then the $\langle m_1, \ldots, m_p \rangle$ notation for λ is $\langle i, j, l \rangle$.

(2) If the m-bead abacus for λ has two beads of weight 1 on runner j and one bead of weight 1 on runner l, then the $\langle m_1, \ldots, m_p \rangle$ notation for λ is $\langle j, j, l \rangle$.

(3) If the m-bead abacus for λ has three beads of weight 1 on runner j, then the $\langle m_1, \ldots, m_p \rangle$ notation for λ is $\langle j, j, j \rangle$.

(4) If the m-bead abacus for gl has a single bead of weight 2 on runner j and a single bead of weight 1 on runner l, then the $\langle m_1, \ldots, m_p \rangle$ notation for λ is $\langle j, l \rangle$.

(5) If the m-bead abacus for λ has one bead of weight 2 and one bead of weight 1 on runner j, then the $\langle m_1, \ldots, m_p \rangle$ notation for λ is $\langle j, j \rangle$.

(6) If the m-bead abacus for λ has a single bead of weight 3 on runner j, then the $\langle m_1, \ldots, m_p \rangle$ notation for λ is $\langle j \rangle$.

Remark. We comment here that while the above (nonstandard) notation is designed specifically for partitions of weight 3, there are analogous (and simpler) definitions valid for partitions of weight 1 and 2 (see, for example, [7, definition 2.1] for the weight 2 case).

Let B be a block of some symmetric group $\mathfrak{S}_n$ of weight ω with core $b = (b_1, \ldots, b_t)$, $b_t > 0$. Let Γ be the $(t + \omega p)$-element β-set for β, and suppose that when Γ is displayed on an abacus with p runners, there are Γ_j beads on runner j and that $\Gamma_i = \Gamma_{i-1} + \kappa$ for some $i \geq 2$ and positive κ. Let $\overline{B}_i$ be the block of $\mathfrak{S}_{n-\kappa}$ of weight ω whose core $\bar{b}$ can be displayed by the $(t + \omega p)$-element β-set $\overline{\Gamma}$ satisfying

$$\begin{aligned} \overline{\Gamma}_j &= \Gamma_j \quad (j \neq i, i-1), \\ \overline{\Gamma}_i &= \Gamma_{i-1}, \\ \overline{\Gamma}_{i-1} &= \Gamma_i. \end{aligned}$$

Such blocks B and $\overline{B}_i$ are said to form a $[\omega{:}\kappa]$ pair. Scopes proved that when $\kappa \geq \omega$, the blocks B and $\overline{B}_i$ are Morita equivalent. In the defect 2 case, a pair of blocks B and $\overline{B}_i$ are either Morita equivalent or form a [2:1] pair (see [11, Section 5]). If a pair of defect 3 blocks are not Morita equivalent, they must form either a [3:1] pair or a [3:2] pair. These pairs are considered in detail in the next two sections. We conclude this section by stating a useful result of James which shows how, in a particular situation, a decomposition number for a symmetric group may be obtained from one of a smaller symmetric group.

THEOREM 1.2 (James [2]). (1) *Suppose that λ and μ are partitions of n with $\lambda_1 = \mu_1 = m$ and λ p-regular. Let $\lambda^R = (\lambda_2, \lambda_3, \ldots)$ and $\mu^R = (\mu_2, \mu_3, \ldots)$. Then the composition multiplicity of D_k^λ in S_k^μ as an $\mathfrak{S}_n$-module equals the composition multiplicity of $D_k^{\lambda^R}$ in $S_k^{\mu^R}$ as an $\mathfrak{S}_{n-m}$-module.*

(2) *Suppose that $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$ are partitions of n with λ_m, $\mu_m \neq 0$ and λ p-regular. Let $\lambda^C = (\lambda_1 - 1, \ldots, \lambda_m - 1)$ and $\mu^C = (\mu_1 - 1, \ldots, \mu_m - 1)$. Then the composition multiplicity of D_k^λ in S_k^μ as an $\mathfrak{S}_n$-module equals the composition multiplicity of $D_k^{\lambda^C}$ in $S_k^{\mu^C}$ as an $\mathfrak{S}_{n-m}$-module.*

2. ON [3:1] PAIRS

We first consider defect 3 blocks which form [3:1] pairs. Such pairs will consist of a block B of $\mathfrak{S}_n$ whose core $b = (b_1, \ldots, b_t)$ has one more bead on runner i than on runner $i-1$ when displayed on an abacus of p runners with $t + 3p$ beads, and a block $\overline{B}_i$ of $\mathfrak{S}_{n-1}$ with core $\bar{b}$ whose $(t + 3p)$-bead abacus display is obtained by interchanging runners i and

$i-1$ of the $(t+3p)$-bead abacus display of b. The first example of a [3:1] pair when k has characteristic 5 is given by the principal blocks of $k\mathfrak{S}_{16}$ and $k\mathfrak{S}_{15}$. See Fig. 1.

Let λ be a partition of n belonging to B. Then the sum of the weights of runners i and $i-1$ of the abacus display of the $(t+\omega p)$-element β-set of λ is called the *mass* of λ. The mass of a partition $\bar{\lambda}$ of $\bar{B}_i$ is defined similarly. Note that the mass of a partition of either block of a $[\omega:\kappa]$ pair lies between 0 and ω. Now suppose that the blocks B and $\bar{B}_i$ form a [3:1] pair. Then there are ten possibilities for the runners i and $i-1$ in a display for a partition with mass 3 belonging to B.

The first four abacus displays shown in Fig. 2 have a unique bead on runner i that can be moved one place to the left onto runner $i-1$. Therefore, if λ in B has a $(t+3p)$-element β-set corresponding to one of these four abacus displays, then there is a unique partition $\bar{\lambda}$ in $\bar{B}_i$ such that $S_k^{\lambda}\downarrow_{\bar{B}_i} \cong S_k^{\bar{\lambda}}$ and $S_k^{\bar{\lambda}}\uparrow^B \cong S_k^{\lambda}$.

The remaining six abacus displays, however, have two beads on runner i that can be moved onto runner $i-1$. Label the partitions corresponding to these abacus displays as follows: Let $\alpha_i, \beta_i, \gamma_i$ denote the partitions $\langle i,i\rangle$, $\langle i,i-1\rangle$, $\langle i-1\rangle$ and let $\alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}$ denote the partitions $\langle i,i,i\rangle, \langle i,i-1,i-1\rangle, \langle i-1,i-1\rangle$. Then if we let $\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i$ denote the partitions $\langle i\rangle$, $\langle i-1,i\rangle$, $\langle i-1,i-1\rangle$ belonging to the block $\bar{B}_i$ and $\bar{\alpha}_{i-1}, \bar{\beta}_{i-1}, \bar{\gamma}_{i-1}$ denote $\langle i,i\rangle, \langle i-1,i,i\rangle, \langle i-1,i-1,i-1\rangle$, we have

$$S_k^{\alpha_c}\downarrow_{\bar{B}_i} \sim S_k^{\bar{\alpha}_c} \oplus S_k^{\bar{\beta}_c}; \qquad S_k^{\bar{\alpha}_c}\uparrow^B \sim S_k^{\alpha_c} \oplus S_k^{\beta_c}, \tag{A$_1$}$$

$$S_k^{\beta_c}\downarrow_{\bar{B}_i} \sim S_k^{\bar{\alpha}_c} \oplus S_k^{\bar{\gamma}_c}; \qquad S_k^{\bar{\beta}_c}\uparrow^B \sim S_k^{\alpha_c} \oplus S_k^{\gamma_c}, \tag{B$_1$}$$

$$S_k^{\gamma_c}\downarrow_{\bar{B}_i} \sim S_k^{\bar{\beta}_c} \oplus S_k^{\bar{\gamma}_c}; \qquad S_k^{\bar{\gamma}_c}\uparrow^B \sim S_k^{\beta_c} \oplus S_k^{\gamma_c}, \tag{C$_1$}$$

where $c=i$ or $i-1$. Note that α_i and $\bar{\alpha}_i$ are always p-regular.

Now consider partitions of mass 2 in a [3:1] pair. There are five possible arrangements of the ith and $(i-1)$th runners. See Fig. 3.

The first and last of the displays in Fig. 3 have a unique bead on runner i that can be moved onto runner $i-1$. Therefore, a partition λ in B with a $(t+3p)$-element β-set corresponding to either display has $S_k^{\lambda}\downarrow_{\bar{B}_i} \cong S_k^{\bar{\lambda}}$

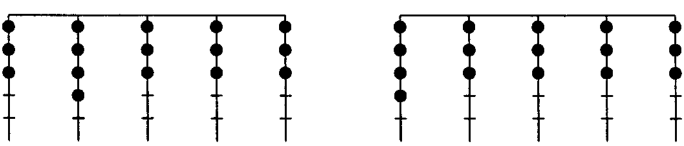


FIG. 1. Abacus displays of the cores of the principal blocks of $k\mathfrak{S}_{16}$ and $k\mathfrak{S}_{15}$.

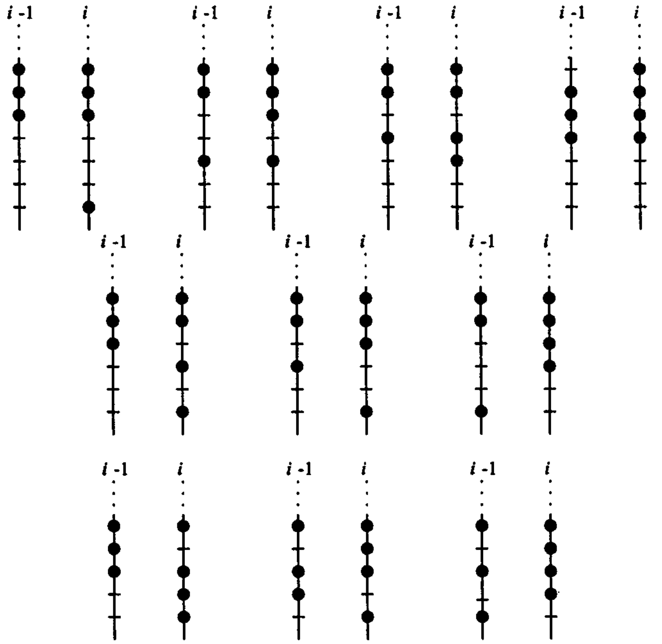


FIG. 2. Abacus displays of partitions of mass 3.

and $S_k^{\bar{\lambda}}\uparrow^B \cong S_k^{\lambda}$ for a unique $\bar{\lambda}$ in $\overline{B}_i$. The remaining three partitions have two movable beads on runner i. So if we let $\alpha_j, \beta_j, \gamma_j$ denote the partitions $\langle i, i, j\rangle, \langle i, i-1, j\rangle, \langle i-1, j\rangle$ of B and let $\bar{\alpha}_j, \bar{\beta}_j, \bar{\gamma}_j$ denote the partitions $\langle i, j\rangle, \langle i, i-1, j\rangle, \langle i-1, i-1, j\rangle$ of $\overline{B}_i$, for $1 \leq j \leq p$, $j \neq i, i-1$, then (A_1), (B_1), and (C_1) above also hold for $c = j$. Note that in all cases, $\alpha_c > \beta_c > \gamma_c$ and $\bar{\alpha}_c > \bar{\beta}_c > \bar{\gamma}_c$.

All the remaining partitions of B are of mass 1 or 0. In either case, there is always a single movable bead on runner i. Hence we have the following result.

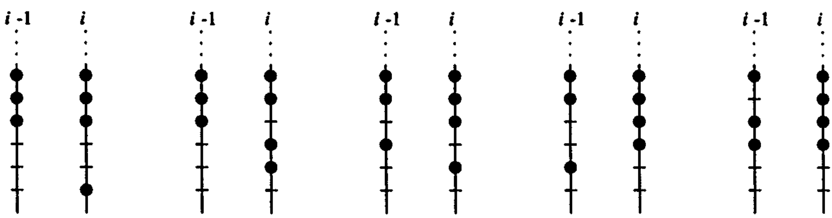


FIG. 3. Abacus displays of partitions of mass 2.

LEMMA 2.1. *Let λ be a partition of n and suppose that S_k^λ belongs to the block B. If λ is not equal to $\alpha_c, \beta_c, \gamma_c$, for any $1 \le c \le p$, then there is a unique partition $\overline{\lambda}$ of $n-1$, which is not equal to $\overline{\alpha}_c$, $\overline{\beta}_c$, or $\overline{\gamma}_c$, such that $S_k^{\overline{\lambda}}$ belongs to the block $\overline{B}_i$ and*

$$S_k^\lambda \downarrow_{\overline{B}_i} \cong S^{\overline{\lambda}_k},$$

$$S^{\overline{\lambda}_k} \uparrow^B \cong S_k^\lambda.$$

This result provides us with a partial correspondence $\lambda \mapsto \overline{\lambda}$ between partitions of B and partitions of $\overline{B}_i$.

DEFINITION 2.2. If λ is a partition whose Specht module belongs to B and if λ is not equal to α_c, β_c, or γ_c, for any $1 \le c \le p$, let $\Phi(\lambda) = \overline{\lambda}$.

The map Φ has the effect of interchanging runners i and $i-1$ of the associated abacus displays and therefore preserves the lexicographic ordering of partitions and p-singularity (see Scopes [11, 12]), from which we deduce the following.

COROLLARY 2.3. *If λ is p-regular and $\lambda \ne \alpha_c$, β_c, or γ_c, $1 \le c \le p$, then*

$$D_k^\lambda \downarrow_{\overline{B}_i} \cong D_k^{\overline{\lambda}},$$

$$D_k^{\overline{\lambda}} \uparrow^B \cong D_k^\lambda.$$

The abacus displays of α_c, β_c, and γ_c have a bead on runner $i-1$ which can be moved one place to the right onto runner i. This action corresponds to inducing the associated Specht modules to a block $\hat{B}$ of defect 1 of $\mathfrak{S}_{n+1}$. The $(t+3p)$-bead abacus display of the core of $\hat{B}$ has one bead less on the runner $i-1$ and one bead more on the runner i than that of the core of B.

There is a unique permutation π of $\{1, \dots, p\}$ so that $\alpha_{\pi(1)} > \alpha_{\pi(2)} > \cdots > \alpha_{\pi(p)}$. Then $S_k^{\alpha_{\pi(c)}} \uparrow^{\hat{B}} \cong S_k^{\langle \pi(c) \rangle}$ with $\langle \pi(1) \rangle > \langle \pi(2) \rangle > \cdots > \langle \pi(p) \rangle$ in $\hat{B}$ (see the remark following Definition 1.1 above). Since the block $\hat{B}$ has defect 1, its decomposition matrix is of the following form:

$$\begin{array}{c} \\ S_k^{\langle\pi(1)\rangle} \\ S_k^{\langle\pi(2)\rangle} \\ S_k^{\langle\pi(3)\rangle} \\ \vdots \\ S_k^{\langle\pi(p-1)\rangle} \\ S_k^{\langle\pi(p)\rangle} \end{array}
\begin{array}{c} \begin{array}{ccccc} D_k^{\langle\pi(1)\rangle} & D_k^{\langle\pi(2)\rangle} & D_k^{\langle\pi(3)\rangle} & \cdots & D_k^{\langle\pi(p-1)\rangle} \end{array} \\
\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 1 & \\ & & & & & 1 \end{pmatrix} \end{array}.$$

Note that $\langle\pi(p)\rangle$ is p-singular. By the abacus displays of $\langle\pi(c)\rangle$ for $1 \le c \le p$, we deduce that

$$S_k^{\langle\pi(c)\rangle}\downarrow_B\uparrow^{\hat{B}} \sim 3S_k^{\langle\pi(c)\rangle},$$

so

$$D_k^{\langle\pi(1)\rangle}\downarrow_B\uparrow^{\hat{B}} \sim 3D_k^{\langle\pi(1)\rangle}.$$

Suppose that, for $1 < l \le p-1$, $D_k^{\langle\pi(l-1)\rangle}\downarrow_B\uparrow^{\hat{B}} \sim 3D_k^{\langle\pi(l-1)\rangle}$. Then,

$$\begin{aligned} S_k^{\langle\pi(l)\rangle}\downarrow_B\uparrow^{\hat{B}} &\sim \left[D_k^{\langle\pi(l)\rangle} + D_k^{\langle\pi(l-1)\rangle}\right]\downarrow_B\uparrow^{\hat{B}} \\ &\sim D_k^{\langle\pi(l)\rangle}\downarrow_B\uparrow^{\hat{B}} + 3D_k^{\langle\pi(l-1)\rangle}. \end{aligned}$$

Hence

$$D_k^{\langle\pi(l)\rangle}\downarrow_B\uparrow^{\hat{B}} \sim 3D_k^{\langle\pi(l)\rangle}.$$

So by induction, $D_k^{\langle\pi(c)\rangle}\downarrow_B\uparrow^{\hat{B}} \sim 3D_k^{\langle\pi(c)\rangle}$ for all $1 \le c \le p-1$. Since by Schaper's formula, $S_k^{\beta_{\pi(c)}}$ and $S_k^{\gamma_{\pi(c)}}$ both contain a copy of $D_k^{\alpha_{\pi(c)}}$, we have $D_k^{\alpha_{\pi(c)}}\uparrow^{\hat{B}} = D_k^{\langle\pi(c)\rangle}$, $D_k^{\beta_{\pi(c)}}\uparrow^{\hat{B}} = D_k^{\gamma_{\pi(c)}}\uparrow^{\hat{B}} = 0$, and $D_k^{\langle\pi(c)\rangle}\downarrow_B$ has three copies of $D_k^{\alpha_{\pi(c)}}$. Also, $D_k^{\langle\pi(c)\rangle}\downarrow_B$ has no copies of $D_k^{\alpha_{\pi(l)}}$, where $l \ne c$; otherwise $D_k^{\langle\pi(c)\rangle}\downarrow_B\uparrow^{\hat{B}}$ would have a copy of $D_k^{\langle\pi(l)\rangle}$. Therefore, from the decomposition matrix of the block $\hat{B}$, we can deduce statement (1) in the following lemma. Statement (2) follows from Theorem 1.2.

LEMMA 2.4. *For* $1 \le c \le p-1$, $D_k^{\alpha_{\pi(c)}}$ *occurs once as a composition factor of the Specht modules* $S_k^{\alpha_{\pi(c)}}$, $S_k^{\beta_{\pi(c)}}$, $S_k^{\gamma_{\pi(c)}}$, $S_k^{\alpha_{\pi(c+1)}}$, $S_k^{\beta_{\pi(c+1)}}$, *and* $S_k^{\gamma_{\pi(c+1)}}$, *and does not occur in any other Specht module.*

(2) $D_k^{\beta_{\pi(c)}}$ *occurs once as a composition factor of* $S_k^{\gamma_{\pi(c)}}$ *if* $\beta_{\pi(c)}$ *is* p*-regular.*

We obtain a similar result below for the irreducibles $D_k^{\overline{\alpha}_{\pi(c)}}$, $1 \le c \le p-1$, of $\overline{B}_i$ by considering restriction to a block $\check{B}$ of defect 1 of $\mathfrak{S}_{n-2}$. The $(t+3p)$-bead abacus display of the core of $\check{B}$ has one bead more on runner $i-1$ and one bead less on runner i than that of $\overline{B}_i$.

LEMMA 2.5. (1) *For* $1 \le c \le p-1$, $D_k^{\overline{\alpha}_{\pi(c)}}$ *occurs once as a composition factor of the Specht modules* $S_k^{\overline{\alpha}_{\pi(c)}}$, $S_k^{\overline{\beta}_{\pi(c)}}$, $S_k^{\overline{\gamma}_{\pi(c)}}$, $S_k^{\overline{\alpha}_{\pi(c+1)}}$, $S_k^{\overline{\beta}_{\pi(c+1)}}$, *and* $S_k^{\overline{\gamma}_{\pi(c+1)}}$, *and does not occur in any other Specht module.*

(2) $D_k^{\overline{\beta}_{\pi(c)}}$ *occurs once as a composition factor of* $S_k^{\overline{\gamma}_{\pi(c)}}$ *if* $\overline{\beta}_{\pi(c)}$ *is* p*-regular.*

Since $\hat{B}$ is of defect 1, the projective indecomposable $P(D_k^{\langle \pi(c) \rangle})$ $\sim \frac{S_k^{\langle \pi(c) \rangle}}{S_k^{\langle \pi(c+1) \rangle}}$. The projective indecomposable $P(D_k^{\alpha_{\pi(c)}})$ of B is given by $P(D_k^{\langle \pi(c) \rangle})\downarrow_B$, so by the Branching Theorem we can add the following corollary.

COROLLARY 2.6.

$$P\left(D_k^{\alpha_{\pi(c)}}\right) \sim \begin{matrix} S_k^{\alpha_{\pi(c)}} \\ S_k^{\beta_{\pi(c)}} \\ S_k^{\gamma_{\pi(c)}} \\ S_k^{\alpha_{\pi(c+1)}} \\ S_k^{\beta_{\pi(c+1)}} \\ S_k^{\gamma_{\pi(c+1)}} \end{matrix}; \qquad P\left(D_k^{\overline{\alpha}_{\pi(c)}}\right) \sim \begin{matrix} S_k^{\overline{\alpha}_{\pi(c)}} \\ S_k^{\overline{\beta}_{\pi(c)}} \\ S_k^{\overline{\gamma}_{\pi(c)}} \\ S_k^{\overline{\alpha}_{\pi(c+1)}} \\ S_k^{\overline{\beta}_{\pi(c+1)}} \\ S_k^{\overline{\gamma}_{\pi(c+1)}} \end{matrix}.$$

We next consider restriction to $\overline{B}_i$ of the irreducible B-modules $D_k^{\alpha_c}$, $D_k^{\beta_c}$, and $D_k^{\gamma_c}$ and induction to B of the irreducible $\overline{B}_i$-modules $D_k^{\overline{\alpha}_c}$, $D_k^{\overline{\beta}_c}$, and $D_k^{\overline{\gamma}_c}$.

LEMMA 2.7. (1) *If α_c is p-regular $D_k^{\alpha_c}\downarrow_{\overline{B}_i}$ is indecomposable with irreducible head and socle $D_k^{\overline{\alpha}_c}$ and $D_k^{\overline{\alpha}_c}\uparrow^B$ is indecomposable with irreducible head and socle $D_k^{\alpha_c}$.*

(2) *If β_c is p-regular, $D_k^{\beta_c}\downarrow_{\overline{B}_i} = D_k^{\overline{\gamma}_c}$ and $D_k^{\overline{\gamma}_c}\uparrow^B = D_k^{\beta_c}$.*

(3) *If γ_c is p-regular, $D_k^{\gamma_c}\downarrow_{\overline{B}_i} = D_k^{\overline{\beta}_c}$ and $D_k^{\overline{\beta}_c}\uparrow^B = D_k^{\gamma_c}$.*

Proof. Suppose $\alpha_{\pi(1)} > \alpha_{\pi(2)} > \cdots > \alpha_{\pi(p)}$. By ($A_1$), $D_k^{\alpha_{\pi(1)}}\downarrow_{\overline{B}_i}$ has two composition factors isomorphic to $D_k^{\overline{\alpha}_{\pi(1)}}$, while $D_k^{\overline{\alpha}_{\pi(1)}}\uparrow^B$ has two composition factors isomorphic to $D_k^{\alpha_{\pi(1)}}$. By Corollary 2.3, all $D_k^{\overline{\lambda}}$ with $\overline{\lambda} > \overline{\alpha}_{\pi(1)}$ induce to give irreducible B-modules D_k^{λ}, where $\lambda \neq \alpha_{\pi(1)}$. Therefore, by Frobenius Reciprocity, $D_k^{\alpha_{\pi(1)}}\downarrow_{\overline{B}_i}$ can only have copies of $D_k^{\overline{\alpha}_{\pi(1)}}$ and $D_k^{\overline{\beta}_{\pi(1)}}$ in its head and socle. Similarly, the head and socle of $D_k^{\overline{\alpha}_{\pi(1)}}\uparrow^B$ can only contain copies of $D_k^{\alpha_{\pi(1)}}$ and $D_k^{\beta_{\pi(1)}}$.

Suppose now that $\beta_{\pi(1)}$ is p-regular and consider $D_k^{\beta_{\pi(1)}}\downarrow_{\overline{B}_i}$. By ($B_1$), $S_k^{\beta_{\pi(1)}}\downarrow_{\overline{B}_i}$ has two copies of $D_k^{\overline{\alpha}_{\pi(1)}}$ and one copy of $D_k^{\overline{\beta}_{\pi(1)}}$. By ($A_1$), however, $D_k^{\alpha_{\pi(1)}}\downarrow_{\overline{B}_i}$ has one copy of $D_k^{\overline{\beta}_{\pi(1)}}$ in addition to two copies of $D_k^{\overline{\alpha}_{\pi(1)}}$. Therefore, $D_k^{\beta_{\pi(1)}}\downarrow_{\overline{B}_i}$ has no copies of $D_k^{\overline{\alpha}_{\pi(1)}}$ or $D_k^{\overline{\beta}_{\pi(1)}}$ and so $D_k^{\beta_{\pi(1)}}\downarrow_{\overline{B}_i} \cong D_k^{\overline{\gamma}_{\pi(1)}}$ and $D_k^{\overline{\alpha}_{\pi(1)}}\uparrow^B$ is indecomposable with head and socle $D_k^{\alpha_{\pi(1)}}$. Similarly, $D_k^{\overline{\beta}_{\pi(1)}}\uparrow^B \cong D_k^{\gamma_{\pi(1)}}$ and $D_k^{\alpha_{\pi(1)}}\downarrow_{\overline{B}_i}$ is indecomposable with head and socle $D_k^{\overline{\alpha}_{\pi(1)}}$. It then follows from ($C_1$) that if $\gamma_{\pi(1)}$ is p-regular, we have $D_k^{\gamma_{\pi(1)}}\downarrow_{\overline{B}_i} \cong D_k^{\overline{\beta}_{\pi(1)}}$ and $D_k^{\overline{\gamma}_{\pi(1)}}\uparrow^B \cong D_k^{\beta_{\pi(1)}}$.

Now suppose that the lemma holds for $c = \pi(1), \pi(2), \ldots, \pi(r-1)$ and consider $D_k^{\alpha_{\pi(r)}}\downarrow_{\overline{B}_i}$. Since all $D_k^{\overline{\lambda}}$ with $\overline{\lambda} > \overline{\alpha}_{\pi(r)}$ induce to give either irreducible B-modules D_k^λ with $\lambda \neq \alpha_{\pi(r)}$, or indecomposable B-modules with head and socle $D_k^{\alpha_{\pi(s)}}$ with $s < r$, it follows by Frobenius Reciprocity that $D_k^{\alpha_{\pi(r)}}\downarrow_{\overline{B}_i}$ can only have copies of $D_k^{\overline{\alpha}_{\pi(r)}}$ and $D_k^{\overline{\beta}_{\pi(r)}}$ in its head and socle. Statement (2), however, follows for $c = \pi(r)$ from (B_1) and so $D_k^{\alpha_{\pi(r)}}\downarrow_{\overline{B}_i}$ is indecomposable with head and socle $D_k^{\overline{\alpha}_{\pi(r)}}$. A similar argument works for $D_k^{\overline{\alpha}_{\pi(r)}}\uparrow^B$ and statement (3) also follows easily for $c = \pi(r)$, thus proving the lemma by induction. ∎

We could now go on to obtain a large amount of information about the modules in the block B, given information about $\overline{B}_i$. In particular, the decomposition matrices of B and $\overline{B}_i$ agree on all rows except for those corresponding to $S_k^{\alpha_c}$, $S_k^{\beta_c}$, and $S_k^{\gamma_c}$ in B and $S_k^{\overline{\alpha}_c}$, $S_k^{\overline{\beta}_c}$, and $S_k^{\overline{\gamma}_c}$. in $\overline{B}_i$.

The lemma just proved provides us with a 1–1 correspondence between the p-regular partitions of B and the p-regular partitions of $\overline{B}_i$; this motivates the next definition.

DEFINITION 2.8. For $\lambda \in B$, let $\Psi(\lambda)$ be the partition of $\overline{B}_i$ where the socle of $D_k^\lambda\downarrow_{\overline{B}_i}$ is $D_k^{\Psi(\lambda)}$. Then $\Psi(\lambda) = \overline{\lambda}$ if $\lambda \neq \alpha_c$, β_c, or γ_c, and $\Psi(\alpha_c) = \overline{\alpha}_c$, $\Psi(\beta_c) = \overline{\gamma}_c$, and $\Psi(\gamma_c) = \overline{\beta}_c$ when α_c, β_c, and γ_c are p-regular.

We have shown that only irreducibles of the form $D_k^{\alpha_c}$ fail to give an irreducible $\overline{B}_i$-module on restriction and that only those of the form $D_k^{\overline{\alpha}_c}$ fail to give an irreducible B-module on induction. Therefore, we may obtain the Ext-quiver of B from that of $\overline{B}_i$ except for the links attached to those nodes representing an irreducible of the form $D_k^{\alpha_c}$. Moreover, by Corollary 2.6, the only nodes which can be linked to the node representing $D_k^{\alpha_{\pi(c)}}$ correspond to irreducible modules which appear either in the second Loewy layer of $S_k^{\alpha_{\pi(c)}}$ or in the first Loewy layer of $S_k^{\beta_{\pi(c)}}$, $S_k^{\gamma_{\pi(c)}}$, $S_k^{\alpha_{\pi(c+1)}}$, $S_k^{\beta_{\pi(c+1)}}$, or $S_k^{\gamma_{\pi(c+1)}}$.

3. ON [3:2] PAIRS

We now turn our attention to defect 3 blocks which form [3:2] pairs. Such pairs will consist of a block B of $\mathfrak{S}_n$ whose core $b = (b_1, \dots, b_t)$ has two more beads on runner i than on runner $i - 1$ when displayed on an abacus of p runners with $t + 3p$ beads, and a block $\overline{B}_i$ of $\mathfrak{S}_{n-2}$ with core $\overline{b}$ whose $(t + 3p)$-bead abacus display is obtained by interchanging runners i and $i - 1$ of the $(t + 3p)$-bead abacus display of b. The first example of a [3:2] pair when k has characteristic 5 is given by the block of $k\mathfrak{S}_{21}$ with core $(5, 1)$ and the principal block of $k\mathfrak{S}_{19}$. See Fig. 4.

Suppose the blocks B and $\overline{B}_i$ form a [3:2] pair. Then Fig. 5 exhibits the four possible arrangements for the runners i and $i - 1$ in a display for a

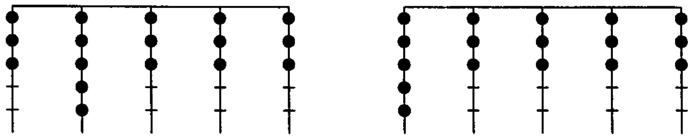


FIG. 4. Abacus displays of the cores (5, 1) and (4).

partition of mass 3 belonging to B with three movable beads on runner i. (Of course, in general, B will contain more partitions of mass 3 than those listed here.)

We label the partitions corresponding to these abacus displays as follows: let α, β, γ, and δ denote the partitions $\langle i,i,i\rangle$, $\langle i,i,i-1\rangle$, $\langle i-1,i\rangle$, and $\langle i-1\rangle$. In each case there are three movable beads on runner i. Restriction to $\overline{B}_i$ corresponds to moving two beads from runner i onto runner $i-1$. Therefore, $S_k^{\lambda}\downarrow_{\overline{B}_i}$, for $\lambda = \alpha$, β, γ, or δ, has six Specht module factors. Let $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$, and $\overline{\delta}$ denote the partitions $\langle i\rangle$, $\langle i,i-1\rangle$, $\langle i,i-1,i-1\rangle$, and $\langle i-1,i-1,i-1\rangle$ in the block $\overline{B}_i$. Then we have

$$S_k^{\alpha}\downarrow_{\overline{B}_i} \sim 2\left(S_k^{\overline{\alpha}} \oplus S_k^{\overline{\beta}} \oplus S_k^{\overline{\gamma}}\right); \qquad S_k^{\overline{\alpha}}\uparrow^{B} \sim 2\left(S_k^{\alpha} \oplus S_k^{\beta} \oplus S_k^{\gamma}\right), \quad (\mathrm{A}_2)$$

$$S_k^{\beta}\downarrow_{\overline{B}_i} \sim 2\left(S_k^{\overline{\alpha}} \oplus S_k^{\overline{\beta}} \oplus S_k^{\overline{\delta}}\right); \qquad S_k^{\overline{\beta}}\uparrow^{B} \sim 2\left(S_k^{\alpha} \oplus S_k^{\beta} \oplus S_k^{\delta}\right), \quad (\mathrm{B}_2)$$

$$S_k^{\gamma}\downarrow_{\overline{B}_i} \sim 2\left(S_k^{\overline{\alpha}} \oplus S_k^{\overline{\gamma}} \oplus S_k^{\overline{\delta}}\right); \qquad S_k^{\overline{\gamma}}\uparrow^{B} \sim 2\left(S_k^{\alpha} \oplus S_k^{\gamma} \oplus S_k^{\delta}\right), \quad (\mathrm{C}_2)$$

$$S_k^{\delta}\downarrow_{\overline{B}_i} \sim 2\left(S_k^{\overline{\beta}} \oplus S_k^{\overline{\gamma}} \oplus S_k^{\overline{\delta}}\right); \qquad S_k^{\overline{\delta}}\uparrow^{B} \sim 2\left(S_k^{\beta} \oplus S_k^{\gamma} \oplus S_k^{\delta}\right). \quad (\mathrm{D}_2)$$

Note that $\alpha > \beta > \gamma > \delta$ and $\overline{\alpha} > \overline{\beta} > \overline{\gamma} > \overline{\delta}$ and that α and $\overline{\alpha}$ are always p-regular.

Each of the remaining partitions of B has at most two movable beads on runner i. We can therefore state a lemma similar to Lemma 2.1.

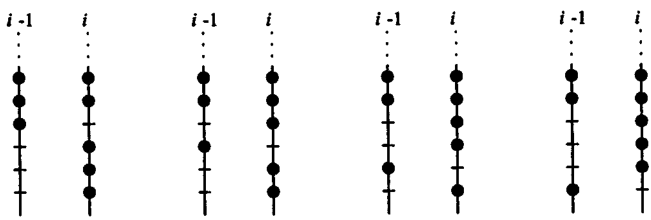


FIG. 5. Abacus displays of partitions of mass 3.

LEMMA 3.1. *Let λ be a partition of n and suppose that S_k^λ belongs to the block B. If λ is not equal to α, β, γ, or δ, then there is a unique partition $\overline{\lambda}$ of $n-2$, which is not equal to $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$, or $\overline{\delta}$, such that $S_k^{\overline{\lambda}}$ belongs to the block $\overline{B}_i$ and*

$$S_k^\lambda \downarrow_{\overline{B}_i} \sim 2S_k^{\overline{\lambda}},$$

$$S_k^{\overline{\lambda}} \uparrow^B \sim 2S_k^\lambda.$$

As in the [3:1] pair case, we obtain a partial correspondence between the partitions of B and $\overline{B}_i$ which preserves p-regularity and lexicographic ordering.

DEFINITION 3.2. If λ is a partition whose Specht module belongs to B and if λ is not equal to α, β, γ, or δ, let $\Phi(\lambda) = \overline{\lambda}$.

COROLLARY 3.3. *If λ is p-regular and $\lambda \neq \alpha$, β, γ, or δ, then*

$$D_k^\lambda \downarrow_{\overline{B}_i} \sim 2D_k^{\overline{\lambda}},$$

$$D_k^{\overline{\lambda}} \uparrow^B \sim 2D_k^\lambda.$$

We now consider the multiplicity of the irreducibles D_k^α, D_k^β, and D_k^γ as composition factors of S_k^β, S_k^γ, and S_k^δ.

LEMMA 3.4. (1) *D_k^α occurs once as a composition factor of S_k^α, S_k^β, S_k^γ, and S_k^δ, and does not occur in any other Specht module.*

(2) *D_k^β occurs once as a composition factor of S_k^γ and S_k^δ if β is p-regular.*

(3) *D_k^γ occurs once as a composition factor of S_k^δ if γ is p-regular.*

Proof. The abacus displays of α, β, γ, and δ each have a unique bead on runner $i-1$ which can be moved one place to the right onto runner i. This action corresponds to inducing the associated Specht modules to a block $\hat{B}$ of defect 0 of $\mathfrak{S}_{n+1}$. The $(t+3p)$-bead abacus display of the single partition, ν say, corresponding to $\hat{B}$ has one bead less on the runner $i-1$ and one bead more on the runner i than that of the core of B. Hence, by the Branching Theorem,

$$S_k^\alpha \uparrow^{\hat{B}} \cong S_k^\nu, \qquad S_k^\beta \uparrow^{\hat{B}} \cong S_k^\nu,$$

$$S_k^\gamma \uparrow^{\hat{B}} \cong S_k^\nu, \qquad S_k^\delta \uparrow^{\hat{B}} \cong S_k^\nu,$$

and $S_k^\lambda \uparrow^{\hat{B}} = 0$ for all $\lambda \neq \alpha$, β, γ, or δ. It follows that S_k^α, S_k^β, S_k^γ, and S_k^δ each have precisely one irreducible composition factor which is not a factor of any Specht module S_k^λ, $\lambda \neq \alpha, \beta, \gamma, \delta$ and which gives a copy of

S_k^ν on induction to $\mathfrak{S}_{n+1}$. In the case of S_k^α it is D_k^α. Since by Schaper's formula, S_k^β, S_k^γ, and S_k^δ all contain copies of D_k^α, statement (1) is proved. Statements (2) and (3) follow from Theorem 1.2. ∎

Again we obtain a similar result for the irreducibles of $\overline{B}_i$ by considering restriction to a block $\check{B}$ of defect 0 of $\mathfrak{S}_{n-3}$. The $(t+3p)$-bead abacus display of the single partition η of $\check{B}$ has one bead more on runner $i-1$ and one bead less on runner i than that of the core of $\overline{B}_i$.

LEMMA 3.5. (1) *$D_k^{\overline{\alpha}}$ occurs once as a composition factor of $S_k^{\overline{\alpha}}$, $S_k^{\overline{\beta}}$, $S_k^{\overline{\gamma}}$, and $S_k^{\overline{\delta}}$, and does not occur in any other Specht module.*

(2) *$D_k^{\overline{\beta}}$ occurs once as a composition factor of $S_{k_\nu}^{\overline{\gamma}}$ and $S_k^{\overline{\delta}}$ if $\overline{\beta}$ is p-regular.*

(3) *$D_k^{\overline{\gamma}}$ occurs once as a composition factor of $S_k^{\overline{\delta}}$ if $\overline{\gamma}$ is p-regular.*

The projective indecomposables $P(D_k^\alpha)$ and $P(D_k^{\overline{\alpha}})$ are given by $S_k^\nu\downarrow_B$ and $S_k^\eta\uparrow^{\overline{B}_i}$. So by the Branching Theorem, we can add the following corollary.

COROLLARY 3.6.

$$P(D_k^\alpha) \sim \begin{matrix} S_k^\alpha \\ S_k^\beta \\ S_k^\gamma \\ S_k^\delta \end{matrix}\ ; \qquad P(D_k^{\overline{\alpha}}) \sim \begin{matrix} S_k^{\overline{\alpha}} \\ S_k^{\overline{\beta}} \\ S_k^{\overline{\gamma}} \\ S_k^{\overline{\delta}} \end{matrix}\ .$$

Now consider the restriction and induction of irreducibles and show that only $D_k^\alpha\downarrow_{\overline{B}_i}$ fails to give a semisimple $\overline{B}_i$-module, and that only $D_k^{\overline{\alpha}}\uparrow^B$ fails to give a semisimple B-module.

LEMMA 3.7. (1) *$D_k^\alpha\downarrow_{\overline{B}_i}$ is the direct sum of at least two indecomposable modules, each with head and socle $D_k^{\overline{\alpha}}$. $D_k^{\overline{\alpha}}\uparrow^B$ is the direct sum of at least two indecomposable modules, each with head and socle D_k^α.*

(2) *If β is p-regular, $D_k^\beta\downarrow_{\overline{B}_i} \sim 2D_k^{\overline{\delta}}$ and $D_k^{\overline{\delta}}\uparrow^B \sim 2D_k^\beta$.*

(3) *If γ is p-regular, $D_k^\gamma\downarrow_{\overline{B}_i} \sim 2D_k^{\overline{\gamma}}$ and $D_k^{\overline{\gamma}}\uparrow^B \sim 2D_k^\gamma$.*

(4) *If δ is p-regular, $D_k^\delta\downarrow_{\overline{B}_i} \sim 2D_k^{\overline{\beta}}$ and $D_k^{\overline{\beta}}\uparrow^B \sim 2D_k^\delta$.*

Proof. The composition factors are determined from (A_2)–(D_2). In particular, $D_k^\alpha\downarrow_{\overline{B}_i}$ has six factors isomorphic to $D_k^{\overline{\alpha}}$ while $D_k^{\overline{\alpha}}\uparrow^B$ has six factors isomorphic to D_k^α. By Corollary 3.6, however, the projective modules $P(D_k^\alpha)$ and $P(D_k^{\overline{\alpha}})$ only have four copies of D_k^α and $D_k^{\overline{\alpha}}$, so both $D_k^\alpha\downarrow_{\overline{B}_i}$ and $D_k^{\overline{\alpha}}\uparrow^B$ must be a direct sum of at least two indecomposables, proving statement (1). ∎

By Lemma 3.1, we know that the decomposition matrices of B and $\overline{B}_i$ agree on all but four rows, and within these rows we now know certain entries.

We will later show that all blocks of defect 3 have the property that they contain no irreducible that extends itself, that is, $(D_k^{\mu}, D_k^{\mu})^1_{k\mathfrak{S}_n} = 0$, for all p-regular μ in a defect 3 block of $\mathfrak{S}_n$. If we assume this is true for $\overline{B}_i$, then all the restricted and induced modules appearing in statements (2)–(4) above and in Corollary 3.3 are semisimple of the form $D_k^{\rho} \oplus D_k^{\rho}$. Therefore, in this case, the Ext-quiver of B can be obtained from that of $\overline{B}_i$ except for the links attached to the node representing D_k^{α}. Moreover, the only nodes which can be linked to the node representing D_k^{α} correspond to irreducibles which appear either in the second Loewy layer of S_k^{α} or in the first Loewy layer of S_k^{β}, S_k^{γ}, or S_k^{δ}.

DEFINITION 3.8. For $\lambda \in B$, Let $\Psi(\lambda)$ be the partition of $\overline{B}_i$ where the socle of $D_k^{\lambda}\downarrow_{\overline{B}_i}$ is a direct sum of copies of $D_k^{\Psi(\lambda)}$. Then $\Psi(\lambda) = \overline{\lambda}$ if $\lambda \neq \alpha, \beta, \gamma,$ or δ, and $\Psi(\alpha) = \overline{\alpha}$, $\Psi(\beta) = \overline{\delta}$, $\Psi(\gamma) = \overline{\gamma}$, and $\Psi(\delta) = \overline{\beta}$ when β, γ, and δ are p-regular.

4. THE PRINCIPAL BLOCK OF $\mathfrak{S}_{3p}$

The principal block of $\mathfrak{S}_{3p}$ is studied in detail in [7], from which we highlight three properties.

THEOREM 4.1. *Over a field of characteristic $p \geq 5$, the principal block of $\mathfrak{S}_{3p}$ has the following properties*:

(1) *All the decomposition numbers are* 0 *or* 1.

(2) $(D_k^{\lambda}, D_k^{\lambda})^1_{k\mathfrak{S}_{3p}} = 0$ *for all p-regular* λ.

(3) $(D_k^{\lambda}, D_k^{\mu})^1_{k\mathfrak{S}_{3p}} = 0$ *or* 1 *for all p-regular λ and* μ.

We aim to show that these three properties hold for all blocks of defect 3. First we examine conditions under which each property would be immediately true for a block B of some symmetric group $\mathfrak{S}_n$ of defect 3, if true for all blocks of defect 3 of smaller symmetric groups. If a block B forms [3:1] or [3:2] pairs with more than one block, we shall distinguish partitions using superscripts. For example, if B forms a [3:1] pair with $\overline{B}_i$, then the partition $\langle i, i, c\rangle$ will be denoted ${}^{i}\alpha_c$, while if B forms a [3:2] pair with $\overline{B}_j$, the partition $\langle j, j, j\rangle$ will be denoted ${}^{j}\alpha$.

(1) *All Decomposition Numbers are* 0 *or* 1

Consider, for example, a block B of $\mathfrak{S}_n$ with core $(b_1, \dots, b_t)$, $b_t > 0$, whose associated $(t + 3p)$-bead abacus display satisfies $\Gamma_i = \Gamma_{i+1} + 1$ and

$\Gamma_j = \Gamma_{j-1} + 2$ for two distinct values $1 \le i, j \le p$. Then B forms a [3:1] pair with a block $\overline{B}_i$ of $\mathfrak{S}_{n-1}$ and a [3:2] pair with a block $\overline{B}_j$ of $\mathfrak{S}_{n-2}$. By restriction to $\overline{B}_i$, we obtain all the rows of the decomposition matrix except for those corresponding to $S_k^{{}^i\alpha_c}$, $S_k^{{}^i\beta_c}$, and $S_k^{{}^i\gamma_c}$, for $1 \le c \le p$. By restriction to $\overline{B}_j$, we obtain all the rows except for those corresponding to $S_k^{{}^j\alpha}$, $S_k^{{}^j\beta}$, $S_k^{{}^i\gamma}$, and $S_k^{{}^j\delta}$. So by combining these results, we can obtain all the rows of the decomposition matrix of B. This can also be done if B forms [3:1] pairs with two blocks of $\mathfrak{S}_{n-1}$ or [3:2] pairs with two blocks of $\mathfrak{S}_{n-2}$ [13]. Hence if the property holds for all blocks of smaller symmetric groups of defect 3, it holds for all B of $\mathfrak{S}_n$ which can be paired with at least two of them.

(2) $(D_k^\lambda, D_k^\lambda)^1_{k\mathfrak{S}_n} = 0$ *for all p-regular* λ

Suppose now that there are at least two blocks $\overline{B}_i$ and $\overline{B}_j$ which form [3:1] or [3:2] pairs with B. Then there are three cases to consider:

(1) If both blocks form [3:1] pairs, we may obtain all the links of the Ext-quiver except for those corresponding to $(D_k^{{}^i\alpha_c}, D_k^{{}^j\alpha_d})^1_{k\mathfrak{S}_n}$, for all $1 \le c, d \le p$.

(2) If both blocks form [3:2] pairs, we may obtain the whole of the Ext-quiver except for the link corresponding to $(D_k^{{}^i\alpha}, D_k^{{}^j\alpha})^1_{k\mathfrak{S}_n}$.

(3) If $\overline{B}_i$ and $\overline{B}_j$ form [3:1] and [3:2] pairs, respectively, we may obtain all the links of the Ext-quiver except for those corresponding to $(D_k^{{}^i\alpha_c}, D_k^{{}^j\alpha})^1_{k\mathfrak{S}_n}$, for all $1 \le c \le p$.

So in each case we can obtain all the self-extensions of B from the Ext-quivers of defect 3 blocks of smaller symmetric groups. In particular, if all defect 3 blocks of smaller symmetric groups have no self-extensions, the same is true for B.

(3) $(D_k^\lambda, D_k^\mu)^1_{k\mathfrak{S}_n} = 0$ *or* 1 *for all p-regular* λ *and* μ

We find from the previous paragraph that if B can be paired with at least two blocks, then most of the Ext-values can be determined. In fact, the remaining Ext-spaces are all zero-dimensional; for example, in (2), if we assume without loss of generality that ${}^i\alpha > {}^j\alpha$, then $D_k^{{}^j\alpha}$ does not appear in $S_k^{{}^i\alpha}$ nor in the first Loewy layer of $S_k^{{}^i\beta}$, $S_k^{{}^i\gamma}$, or $S_k^{{}^i\delta}$, so $D_k^{{}^i\alpha}$ does not extend $D_k^{{}^j\alpha}$. Therefore, if the property holds for all blocks of defect 3 of smaller symmetric groups, it also holds for blocks of $\mathfrak{S}_n$ which can be paired with at least two blocks.

It follows that there are only two types of blocks of $\mathfrak{S}_n$ we need to consider: blocks which form a single [3:1] pair with a block of $\mathfrak{S}_{n-1}$ but form no [3:2] pair with any block of $\mathfrak{S}_{n-2}$, and blocks which form a single

[3:2] pair but no [3:1] pairs. We have observed that if the blocks B and $\overline{B}_i$ form a [3:1] pair, we can obtain the Ext-quiver of B from that of $\overline{B}_i$ except for the links attached to the nodes representing an irreducible of the form $D_k^{\alpha_c}$. Hence multiple and self-extensions can only occur at these nodes if it is assumed that there are no multiple or self-extensions occurring in the Ext-quiver of $\overline{B}_i$. If B and $\overline{B}_i$ form a [3:2] pair, the multiple and self-extensions can only occur at the node representing D_k^{α} in the Ext-quiver of B if there are no multiple or self-extensions in the Ext-quiver of $\overline{B}_i$. If we assume that all decomposition numbers are 0 or 1, then from the structures of the projective indecomposables given in Lemmas 2.6 and 3.6, we deduce that multiple and self-extensions can only occur when some β_c or γ_c (in the [3:1] pair case) or β, γ, or δ (in the [3:2] pair case) is p-singular. These cases are dealt with by appealing to conjugate blocks in order to obtain further information about the projective indecomposables (see Section 6). We will therefore concentrate on proving that all decomposition numbers of defect 3 blocks are 0 or 1.

5. DECOMPOSITION NUMBERS

5.1. *Blocks with rectangular cores*

In this section, we consider a block B that only forms a single [3:1] pair with a block $\overline{B}_i$ of $\mathfrak{S}_{n-1}$ and no [3:2] pairs with any blocks of $\mathfrak{S}_{n-2}$. In this case, there is precisely one runner which has one more bead than the previous runner in the $(t+3p)$-bead abacus display of the core of B. As there are always three beads on the first runner of such a display, this implies that there are at most four beads on any runner. Hence the core of B is rectangular of the form $(i-1)^z$, with $i+z-1 \leq p$. The abacus display has three beads on the first $i-1$ runners, four beads on the next z runners, and three beads on the final $p-i-z+1$ runners. See Fig. 6.

Any two adjacent runners have either the same number of beads, or one runner has four beads and the other has three beads. Given a display for a partition λ of B, the action of moving a bead from a runner s onto the

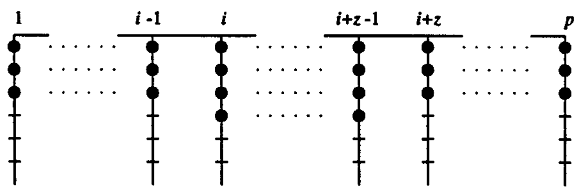


FIG. 6. Display of a rectangular core.

adjacent runner $s-1$ corresponds to restricting to a block of $\mathfrak{S}_{n-1}$. Our approach will be to consider restriction of irreducibles to blocks of $\mathfrak{S}_{n-1}$ of lower defect.

Restriction of irreducibles

Suppose we have a partition μ of B which satisfies the following two conditions: the abacus display has a bead on a runner $s \neq 1, i, i+z$ which is movable one space to the left onto runner $s-1$, and the sum of the weights of the beads on runner s is greater than the sum of the weights of the beads on runner $s-1$. Then the two runners have the same number of beads and the action of moving the bead from runner s to runner $s-1$ corresponds to restricting S_k^{μ} to a block B_s of defect 2. There is always at most one movable bead on runner s, so we obtain a single Specht module $S_k^{\tilde{\mu}}$ on restriction to B_s. Furthermore, for any $\tilde{\mu}$ of B_s, there are precisely two partitions, $\mu_1 > \mu_2$ say, of B such that $S_k^{\mu_1}\downarrow_{B_s} = S_k^{\mu_2}\downarrow_{B_s} = S_k^{\tilde{\mu}}$. By the second condition on μ, we must have $\mu = \mu_1$. The mapping $\mu \mapsto \tilde{\mu}$ of partitions of B satisfying the two conditions above to partitions of B_s preserves lexicographic ordering and p-regularity. As a consequence, $D_k^{\mu}\downarrow_{B_s} = D_k^{\tilde{\mu}}$ (the details are given in [7, Section 3]).

Now suppose μ has a bead on runner 1 (at position mp) movable onto runner p (at position $mp-1$) and that there are no beads on runner p at a higher position $m'p-1$ $(m' > m)$. Then the action of moving the bead from runner 1 to runner p corresponds to restricting S_k^{μ} to a block B_1 of defect 1 or 0. (The defect of B_1 is 0 if $i+z-1=p$; 1 otherwise.) There is at most one movable bead so we obtain a single Specht module $S_k^{\tilde{\mu}}$ on restriction to B_1, but there are four partitions $\mu_1 > \mu_2 > \mu_3 > \mu_4$ belonging to B with $S_k^{\mu_j}\downarrow_{B_1} = S_k^{\tilde{\mu}}$ $(j = 1, 2, 3, 4)$. Their abacus arrays are obtained from the $(t+3p)$-bead display of the core associated with B by sliding up beads on runners 1 and p in four obvious ways. So by the second condition on μ, we have $\mu = \mu_1$, the map $\mu \mapsto \tilde{\mu}$ preserves lexicographic ordering and p-regularity, and $D_k^{\mu}\downarrow_{B_1} = D_k^{\tilde{\mu}}$.

If μ has a bead on runner $i+z$ movable onto runner $i+z-1$ and there are no beads on runner $i+z-1$ at a higher position, then the action of moving the bead corresponds to restricting S_k^{μ} to a block B_{i+z} of defect 1. Again there is at most one movable bead so $S_k^{\mu}\downarrow_{B_{i+z}} = S_k^{\tilde{\mu}}$. There are three partitions $\mu_1 > \mu_2 > \mu_3$ whose corresponding Specht modules restrict to B_{i+z} to give $S_k^{\tilde{\mu}}$ so $\mu = \mu_1$ and $D_k^{\mu}\downarrow_{B_{i+z}} = D_k^{\tilde{\mu}}$ as before.

Most partitions of B will satisfy the criteria given in at least one of the three previous paragraphs and so we deduce that most irreducibles of B give an irreducible on restriction to some block of lower defect. If $D_k^{\mu}\downarrow_{B_s} = D_k^{\tilde{\mu}}$, where B_s has defect 2, 1, or 0, then the multiplicity of D_k^{μ}

in S_k^λ is equal to the multiplicity of $D_k^{\tilde{\mu}}$ in $S_k^\lambda\downarrow_{B_s}$. This value cannot be greater than 1 since all decomposition numbers of blocks of defect 2, 1, or 0 are either 0 or 1. Therefore it just remains to investigate the multiplicities of those irreducibles of B whose partitions do not satisfy the criteria given in any of the three previous paragraphs. Moreover, by Lemma 2.1, these multiplicities cannot be greater than 1 except in the Specht modules $S_k^{\alpha_c}$, $S_k^{\beta_c}$, and $S_k^{\gamma_c}$, for $1 \le c \le p$, so we need only consider such irreducibles D_k^μ with $\mu > \gamma_{i-1}$, $\mu \neq \alpha_c, \beta_c, \gamma_c$. (The partition γ_{i-1} appears further down than any other γ_c in the lexicographic ordering, so no D_k^μ with $\mu < \gamma_{i-1}$ can appear in any $S_k^{\alpha_c}$, $S_k^{\beta_c}$, or $S_k^{\gamma_c}$. Also the multiplicities of irreducibles of the form $D_k^{\alpha_c}$, $D_k^{\beta_c}$, and $D_k^{\gamma_c}$ are dealt with in Lemma 2.4.) A complete list of the remaining (p-regular) partitions is given in Table 1.

Assuming all the decomposition number in the block $\overline{B}_i$ are either 0 or 1, the multiplicity of any D_k^μ as a composition factor of some $S_k^{\alpha_c}$, $S_k^{\beta_c}$, or $S_k^{\gamma_c}$ is at most 2. Moreover, if it is 2, then one of the following holds:

(1) D_k^μ occurs with multiplicity 2 in $S_k^{\alpha_c}$ and multiplicity 1 in both $S_k^{\beta_c}$ and $S_k^{\gamma_c}$, and $D_k^{\overline{\mu}}$ occurs with multiplicity 1 in both $S_k^{\overline{\alpha}_c}$ and $S_{ck}^{\overline{\beta}}$.

(2) D_k^μ occurs with multiplicity 2 in $S_k^{\beta_c}$ and multiplicity 1 in both $S_k^{\alpha_c}$ and $S_k^{\gamma_c}$, and $D_k^{\overline{\mu}}$ occurs with multiplicity 1 in both $S_k^{\overline{\alpha}_c}$ and $S_k^{\overline{\gamma}_c}$.

(3) D_k^μ occurs with multiplicity 2 in $S_k^{\gamma_c}$ and multiplicity 1 in both $S_k^{\alpha_c}$ and $S_k^{\beta_c}$, and $D_k^{\overline{\mu}}$ occurs with multiplicity 1 in both $S_k^{\overline{\beta}_c}$ and $S_k^{\overline{\gamma}_c}$.

(4) D_k^μ occurs with multiplicity 2 in $S_k^{\alpha_c}$, $S_k^{\beta_c}$, and $S_k^{\gamma_c}$, for some c, and $D_k^{\overline{\mu}}$ occurs with multiplicity 1 in $S_k^{\overline{\alpha}_c}$, $S_k^{\overline{\beta}_c}$, and $S_k^{\overline{\gamma}_c}$.

We therefore consider the multiplicities of $D_k^{\overline{\mu}}$ in $S_k^{\overline{\alpha}_c}$, $S_k^{\overline{\beta}_c}$, and $S_k^{\overline{\beta}_c}$ in $\overline{B}_i$. We do this by looking at two sequences of blocks.

TABLE 1

μ	Conditions on i, z
$\langle i\rangle$	—
$\langle i, i+1\rangle$	$i \neq p$; $z \ge 2$
$\langle i, i+z\rangle$	$i+z \le p$
$\langle i+2, i+1, i\rangle$	$i \le p-2$; $z \ge 3$
$\langle i+z, i+1, i\rangle$	$i+z \le p$; $z \ge 2$
$\langle i+1, i\rangle$	$z = 1$; $i \neq p$
$\langle 1, i\rangle$	$i+z = p+1$
$\langle i+z+1, i+z, i\rangle$	$i+z \le p-1$
$\langle i+z, i, 1\rangle$	$i+z \le p$

Sequences of blocks

Suppose $i+z \le p$ and that μ appears in Table 1 and has a bead of nonzero weight on runner $i+z$ and let $\mathfrak{B}_u$ denote the block of $\mathfrak{S}_{3p+(i-2)z+u}$ with core $((i-1)^u,(i-2)^{z-u})$, where $0 \le u < z$. Then we have a sequence of blocks $\mathfrak{B}_0,\dots,\mathfrak{B}_{z-1}$, in which $\mathfrak{B}_0$ has a rectangular core $((i-2)^z)$, $\overline{B}_i = \mathfrak{B}_{z-1}$, and for each j the blocks $\mathfrak{B}_{j+1}$ and $\mathfrak{B}_j$ form a [3:1] pair. Let Φ_j be the map $\lambda \mapsto \overline{\lambda}$ of partitions of $\mathfrak{B}_{j+1}$ to partitions of $\mathfrak{B}_j$ described in Definition 2.2. Then the partitions $\overline{\mu}$ and $\overline{\gamma}_c$ have well-defined images, $\tilde{\mu}$ and $\tilde{\gamma}_c$ say, under the map $\Phi_0 \cdot \Phi_1 \dots \Phi_{z-2}$. The multiplicity of $D_k^{\overline{\mu}}$ in $S_k^{\overline{\gamma}_c}$ in $\overline{B}_i$ is the same as the multiplicity of $D_k^{\tilde{\mu}}$ in $S_k^{\tilde{\gamma}_c}$ in $\mathfrak{B}_0$. The abacus display of $\tilde{\mu}$, however, has only three beads on runners $i+z$ and $i+z-1$ with no beads of nonzero weight on runner $i+z-1$, so we may move the bead of nonzero weight on runner $i+z$ one space to the left. As before, this corresponds to restricting to a block $\mathfrak{B}$ of defect 2 with $D_k^{\tilde{\mu}}\downarrow_{\mathfrak{B}}$ irreducible. Since the multiplicity of $D_k^{\tilde{\mu}}$ in some S_k^{ρ} is equal to the multiplicity of $D_k^{\tilde{\mu}}\downarrow_{\mathfrak{B}}$ in $S_k^{\rho}\downarrow_{\mathfrak{B}}$, $D_k^{\tilde{\mu}}$ can only appear in those $S_k^{\tilde{\gamma}_c}$ which have a nonzero restriction to $\mathfrak{B}$. The image $\tilde{\gamma}_c$ (written in $\langle 3^{i-2},4^z,3^{p-i-z+2}\rangle$ notation) varies as follows:

$$\tilde{\gamma}_c = \begin{cases} \langle i-1, i-1\rangle, & \text{if } c = i;\\ \langle c-1, i-1, i-1\rangle, & \text{if } i+1 \le c \le i+z-1;\\ \langle c, i-1, i-1\rangle, & \text{otherwise.}\end{cases}$$

So only $S_k^{\tilde{\gamma}_{i+z}}$ has a nonzero restriction to $\mathfrak{B}$ and therefore the multiplicity of $D_k^{\overline{\mu}}$ in $S_k^{\overline{\gamma}_c}$ is zero unless $c = i+z$. The multiplicity of $D_k^{\overline{\mu}}$ in $S_k^{\overline{\beta}_{i+z}}$ is also zero since $\overline{\beta}_{i+z}$ has a well-defined image $\tilde{\beta}_{i+z} = \langle i+z, i+z-1, i-1\rangle$ and $S_k^{\tilde{\beta}_{i+z}}\downarrow_{\mathfrak{B}} = 0$. We deduce that $D_k^{\overline{\mu}}$ can never occur with multiplicity 1 in both $S_k^{\overline{\beta}_c}$ $S_k^{\overline{\gamma}_c}$. Hence (3) and (4) above cannot occur and the multiplicity of D_k^{μ} in $S_k^{\gamma_c}$ is at most 1.

We can now eliminate the possibility of D_k^{μ} occurring with multiplicity 2 in any $S_k^{\alpha_c}$, $S_k^{\beta_c}$, and $S_k^{\gamma_c}$ for several cases in Table 1. First consider the two cases $\mu = \langle i+z+1, i+z, i\rangle$ where $i+z \le p-1$, and $\mu = \langle i+z, i, 1\rangle$ where $i+z \le p$. Then if $i \le c \le i+z-1$, μ appears further down the lexicographic ordering than α_c, β_c, and γ_c, so D_k^{μ} cannot appear in $S_k^{\alpha_c}$, $S_k^{\beta_c}$, and $S_k^{\gamma_c}$. For other values of c, that is, $1 \le c \le i-1$ and $i+z \le c \le p$, the partitions μ, α_c, and β_c each have a bead at position $4p+i-1$ (on runner i) and no bead at a higher position. Therefore by Theorem 1.2, the multiplicities of D_k^{μ} in $S_k^{\alpha_c}$ and $S_k^{\beta_c}$ are given by a decomposition matrix of a block of defect 2 and so cannot be greater than 1. Since we have seen above that the multiplicity of D_k^{μ} in any $S_k^{\gamma_c}$ is also at most 1, these two cases can be eliminated. If we allow $i+z = p+1$ and interpret runner

$i + z$ as runner 1 in the above argument, we can also remove the case $\mu = \langle 1, i \rangle$.

Now suppose that, in addition to having a bead of nonzero weight on runner $i + z$, that μ appears in Table 1 and also has a bead of nonzero weight on runner i and let $\mathfrak{B}_v$ be the block of $\mathfrak{S}_{3p+(i-1)(z-1)+v}$ with core $((i-1)^{z-1}, v)$, where $0 \le v < i - 1$. Then we have a second sequence $\mathfrak{B}_0, \dots, \mathfrak{B}_{i-2}$, in which $\mathfrak{B}_0$ has a rectangular core $((i-1)^{z-1})$, $\overline{B}_i = \mathfrak{B}_{i-2}$, and for each j the blocks $\mathfrak{B}_{j+1}$ and $\mathfrak{B}_j$ form a [3:1] pair. Let Θ_j denote the map $\lambda \mapsto \overline{\lambda}$ of partitions of $\mathfrak{B}_{j+1}$ to partitions of $\mathfrak{B}_j$ as in Definition 2.2. Then the partitions $\overline{\mu}$ and $\overline{\alpha}_c$ have well-defined images, $\breve{\mu}$ and $\breve{\alpha}_c$, under the map $\Theta_0 \cdot \Theta_1 \cdots \Theta_{i-3}$. The image $\breve{\alpha}_c$ (written in $\langle 4, 3^{i-1}, 4^{z-1}, 3^{p-i-z+1} \rangle$ notation) varies as follows:

$$\breve{\alpha}_c = \begin{cases} \langle i, c+1 \rangle, & \text{if } c \le i-2; \\ \langle i, i \rangle, & \text{if } c = i-1; \\ \langle i \rangle, & \text{if } c = i; \\ \langle i, c \rangle, & \text{if } c \ge i+1. \end{cases}$$

Since $i + z \le p$, $S_k^{\breve{\alpha}_p}$ has a nonzero restriction to a block $\mathfrak{B}$ of defect 2 corresponding to the action of moving a bead on runner 1 onto runner p, while $\overline{\beta}_p$ has a well-defined image $\breve{\beta}_p = \langle p, i, 1 \rangle$ and $S_k^{\breve{\beta}_p}\downarrow_{\mathfrak{B}} = 0$. If $c \ne p$, however, then $S_k^{\breve{\alpha}_c}\downarrow_{\mathfrak{B}} = 0$. So, when $D_k^{\breve{\mu}}\downarrow_{\mathfrak{B}}$ is nonzero, $D_k^{\breve{\mu}}$ can never occur with multiplicity 1 in both $S_k^{\breve{\alpha}_c}$ and $S_k^{\breve{\beta}_c}$, which in turn implies that $D_k^{\overline{\mu}}$ can never occur with multiplicity 1 in both $S_k^{\overline{\alpha}_c}$ and $S_k^{\overline{\beta}_c}$. Hence in this case, (1) and (4) cannot occur and the multiplicity of D_k^{μ} in $S_k^{\alpha_c}$ is at most 1.

Now consider the following cases: $\mu = \langle i, i+z \rangle$ where $i + z \le p - 1$, $\mu = \langle i+z, i+1, i \rangle$ where $i + z \le p - 1$, $z \ge 2$, and $\mu = \langle i+1, i \rangle$ where $z = 1$, $i \le p - 2$. Then $\overline{\mu}$ has an image $\breve{\mu}$ such that $D_k^{\breve{\mu}}\downarrow_{\mathfrak{B}}$ is nonzero. So the multiplicity of D_k^{μ} in $S_k^{\alpha_c}$ is at most 1. Since μ has a bead of nonzero weight on runner $i + z$, the multiplicity of D_k^{μ} in $S_k^{\gamma_c}$ is also at most 1. Furthermore, since $i + z \ne p$, the multiplicities of $D_k^{\tilde{\mu}}$ in $S_k^{\overline{\alpha}_c}$ and $S_k^{\overline{\gamma}_c}$ cannot both equal 1. This eliminates the possibility of (2) occurring. So D_k^{μ} cannot occur with multiplicity greater than 1 in any Specht module in these cases.

So we have eliminated a number of cases from Table 1. For information on the remaining cases (listed below in Table 2) we look to the conjugate blocks.

Conjugate blocks

In general, if B and $\overline{B}$ form a $[\omega : \kappa]$ pair, then so do the conjugate blocks B' and $\overline{B}'$. Hence the conjugates of a [3:1] pair also form a [3:1]

TABLE 2

μ	Conditions on i, z
$\langle i \rangle$	—
$\langle i, i+1 \rangle$	$i \neq p$; $z \geq 2$
$\langle i, p \rangle$	$i + z = p$
$\langle i+2, i+1, i \rangle$	$i \leq p-2$; $z \geq 3$
$\langle p, i+1, i \rangle$	$i + z = p$; $z \geq 2$
$\langle p, p-1 \rangle$	$z = 1$; $i = p - 1$

pair. Now suppose that, as before, we move from the $(t+3p)$-bead abacus display of the core of B to that of $\overline{B}$ by moving a bead from runner i onto runner $i-1$ and that we obtain the display of the core of $\overline{B}'$ from that of B' by moving a bead from runner j onto runner $j-1$. Let ${}^i\alpha_c$, ${}^i\beta_c$, and ${}^i\gamma_c$ be the usual partitions of B. Then ${}^i\alpha_b'$, ${}^i\beta_c'$, and ${}^i\gamma_c'$ are of the form ${}^j\gamma_d$, ${}^j\beta_d$, and ${}^j\alpha_d$, respectively (for some $1 \leq d \leq p$). In other words, by taking conjugates, we obtain a bijection between the sets $\{{}^i\alpha_c, {}^i\beta_c, {}^i\gamma_c | 1 \leq c \leq p\}$ of B and $\{{}^j\alpha_d, {}^j\beta_d, {}^j\gamma_d | 1 \leq d \leq p\}$ of B'. Also in this notation, $D_k^{{}^i\alpha_c} \otimes alt$ is of the form $D_k^{{}^j\alpha_h}$ for some $1 \leq h \leq p$.

Now suppose $\lambda = {}^i\alpha_c$, ${}^i\beta_c$, or ${}^i\gamma_c$ for some c and consider what this tells us about composition factors of S_k^λ in the block B with rectangular core $(i-1)^z$. The conjugate $S_k^{\lambda'}$ of B' is of the form $S_k^{{}^{z+1}\alpha_d}$, $S_k^{{}^{z+1}\beta_d}$, or $S_k^{{}^{z+1}\gamma_d}$ and the multiplicity of D_k^μ in S_k^λ is equal to the multiplicity of $D_k^{\mu^*}$ $(= D_k^\mu \otimes alt)$ in $S_k^{\lambda'}$. In some cases, the multiplicity of $D_k^{\mu^*}$ in $S_k^{\lambda'}$ is easier to compute than the multiplicity of D_k^μ in S_k^λ. In particular, in the case of $\mu^* < {}^{z+1}\gamma_z$, the irreducible $D_k^{\mu^*}$ cannot occur in $S_k^{\lambda'}$, which implies the multiplicity of D_k^μ in S_k^λ is zero. It is therefore useful in some cases to identify the module $D_k^{\mu^*}$ in order to establish the multiplicities of D_k^μ in the Specht modules $S_k^{{}^i\alpha_c}$, $S_k^{{}^i\beta_c}$, and $S_k^{{}^i\gamma_c}$, for $1 \leq c \leq p$.

Let $\mathfrak{B}_{(i-1)u+v}$ be the block of $\mathfrak{S}_{3p+(i-1)u+v}$ $(0 \leq v < i-1)$ with core $((i-1)^u, v)$. Then we have a sequence of blocks $\mathfrak{B}_0, \ldots, \mathfrak{B}_{(i-1)z}$, in which $\mathfrak{B}_0$ is the principal block of $\mathfrak{S}_{3p}$, $\mathfrak{B}_{(i-1)z} = B$, and where, for each j, the blocks $\mathfrak{B}_{j+1}$ and $\mathfrak{B}_j$ form a [3:1] pair. Let Ψ_j denote the 1–1 map of p-regular partitions of $\mathfrak{B}_{j+1}$ to p-regular partitions of $\mathfrak{B}_j$ defined in Definition 2.8. By taking conjugate blocks, we obtain a similar sequence $\mathfrak{B}'_0, \ldots, \mathfrak{B}'_{(i-1)z}$, in which $\mathfrak{B}'_{(i-1)z} = B'$, $\mathfrak{B}'_{(i-1)u+v}$ has core $((i-1)^u, v)'$, and Ψ_j' is a map $\rho \mapsto \bar{\rho}$ of partitions of $\mathfrak{B}'_{j+1}$ to partitions of $\mathfrak{B}'_j$. Since the actions of restriction and conjugate commute, we have the following commutative diagram for each pair of blocks $\mathfrak{B}_{j+1}$ and $\mathfrak{B}_j$:

$$\begin{array}{ccc} D_k^{\lambda} & \longrightarrow & D_k^{\Psi(\lambda)} \\ \downarrow \otimes \mathrm{alt} & & \downarrow \otimes \mathrm{alt} \\ D_k^{\lambda^*} & \longrightarrow & D_k^{\Psi(\lambda)^*} \end{array}.$$

Therefore, we can use information about the effect of tensoring with *alt* in $\mathfrak{B}_0$ (obtained from [7]) to determine effects of tensoring with *alt* in $\mathfrak{B}_1, \ldots, \mathfrak{B}_{(i-1)z}$. For example, suppose $z = 1$ and $\mu = \langle i \rangle$ of B in $\langle 3^{i-1}, 4, 3^{p-i} \rangle$ notation. Then

$$\begin{aligned} \Psi_0 \cdot \Psi_1 \cdots \Psi_{i-2}(\mu) &= \langle 1 \rangle \text{ in } \langle 4, 3^{p-1} \rangle \text{ notation}, \\ &= \langle p \rangle \text{ in } \langle 3^p \rangle \text{ notation}. \end{aligned}$$

Given that

$$\begin{aligned} \langle p \rangle^* &= \langle 4, 3, 2 \rangle \text{ in } \langle 3^p \rangle \text{ notation}, \\ &= \langle i+3, i+2, i+1 \rangle \text{ in } \langle 4^{i-1}, 3^{p-i+1} \rangle \text{ notation}, \end{aligned}$$

we map apply the map $(\Psi'_{i-2})^{-1} \cdots (\Psi'_1)^{-1} \cdot (\Psi'_0)^{-1}$, where $(\Psi'_j)^{-1}$ is the inverse map of Ψ'_j, to $\langle i+3, i+2, i+1 \rangle$ in $\langle 4^{i-1}, 3^{p-i+1} \rangle$ notation to obtain μ^* of B' in $\langle 3, 4^{i-1}, 3^{p-i} \rangle$ notation as follows:

$$\mu^* = \begin{cases} \langle i+3, i+2, i+1 \rangle, & \text{if } i \leq p-3; \\ \langle p, p-1, 2 \rangle, & \text{if } i = p-2; \\ \langle p, 3, 2 \rangle, & \text{if } i = p-1; \\ \langle 4, 3, 2 \rangle, & \text{if } i = p. \end{cases}$$

Repeating this procedure for all the cases μ appearing in Table 2, we obtain the following list (Table 3) for μ^*. Each case is consistent with the Mullineux map of partitions. If, for a certain case, we can show that the irreducible $D_k^{\mu^*}$ can never occur in any $S_k^{{}^{z+1}\alpha_d}$, $S_k^{{}^{z+1}\beta_d}$, or $S_k^{{}^{z+1}\gamma_d}$ of B', it will follow that the multiplicity of D_k^{μ} in any $S_k^{{}^{i}\alpha_c}$, $S_k^{{}^{i}\beta_c}$, or $S_k^{{}^{i}\gamma_c}$ is zero. This is immediately true if $\mu^* < {}^{z+1}\gamma_z$, and these cases are labelled with † in Table 3. It is also true if μ^* appears in Table 1 (with i replaced by $z+1$ since μ^* belongs to the conjugate block with core z^{i-1}) but not in Table 2. These cases are labelled with ‡ in Table 3. The remaining cases are dealt with by Schaper's formula. (An example is given in the Appendix.)

TABLE 3
Mullineux Mapping of Irreducibles in Table 2

$\mu = \langle i \rangle$	
Conditions on i, z	μ^*
$z = 1;\ i \leq p - 3$	$\langle i + 3, i + 2, i + 1 \rangle$†
$z = 1;\ i = p - 2$	$\langle o, o - 1, 2)$‡
$z = 1;\ i = p - 1$	$\langle p, 3, 2 \rangle$
$z = 1;\ i = p$	$\langle 4, 3, 2 \rangle$
$z = 2;\ i \leq p - 3$	$\langle i + 3, i + 2, 1 \rangle$†
$z = 2;\ i = p - 2$	$\langle 1, p \rangle$†
$z = 2;\ i = p - 1$	$\langle 1, 3 \rangle$‡
$z \geq 3;\ i + z \leq p$	$\langle i + z, 2, 1 \rangle$†
$z \geq 3;\ i + z = p + 1$	$\langle 1, 2 \rangle$†
$\mu = \langle i, i + 1 \rangle$	
Conditions on i, z	μ^*
$z = 2;\ i \leq p - 4$	$\langle i + 4, i + 3, i + 2 \rangle$†
$z = 2;\ i = p - 3$	$\langle p, p - 1, 3 \rangle$‡
$z = 2;\ i = p - 2$	$\langle p, 4, 3 \rangle$
$z = 2;\ i = p - 1$	$\langle 5, 4, 3 \rangle$
$z \geq 3;\ i + z \leq p - 1$	$\langle i + z + 1, i + z, 1 \rangle$†
$z \geq 3;\ i + z = p$	$\langle 1, p \rangle$†
$z \geq 3;\ i + z = p + 1$	$\langle 1, z + 1 \rangle$‡
$\mu = \langle i, p \rangle$	
Conditions on i, z	μ^*
$i + z = p$	$\langle p, z + 1, 1 \rangle$‡
$\mu = \langle i + 2, i + 1, i \rangle$	
Conditions on i, z	μ^*
$3 \leq z \leq p - 3;\ i + z \leq p - 2$	$\langle i + z + 2, i + z + 1, i + z \rangle$†
$3 \leq z \leq p - 3;\ i + z = p - 1$	$\langle p, p - 1, z + 1 \rangle$‡
$3 \leq z \leq p - 3;\ i + z = p$	$\langle p, z + 2, z + 1 \rangle$
$3 \leq z \leq p - 3;\ i + z = p + 1$	$\langle z + 3, z + 2, z + 1 \rangle$
$z = p - 2;\ i = 2$	$\langle p, p - 1 \rangle$
$z = p - 2;\ i = 3$	$\langle p - 1, p \rangle$
$z = p - 1;\ i = 2$	$\langle p \rangle$
$\mu = \langle p, i + 1, i \rangle$	
Conditions on i, z	μ^*
$i + z = p;\ i \geq 4;\ z \geq 2$	$\langle z + 3, z + 2, z + 1 \rangle$
$z = p - 3;\ i = 3$	$\langle p - 2, p - 1 \rangle$
$z = p - 2;\ i = 2$	$\langle p - 1 \rangle$
$\mu = \langle p, p - 1 \rangle$	
Conditions on i, z	μ^*
$z = 1;\ i = p - 1$	$\langle 4, 3, 2 \rangle$

5.2. *Blocks with birectangular cores*

We now consider blocks B which form a single [3:2] pair with a block $\overline{B}_i$ of $\mathfrak{S}_{n-2}$ and no [3:1] pairs with any blocks of $\mathfrak{S}_{n-1}$. In this case, there is precisely one runner which has two more beads than the previous runner in the $(t+3p)$-bead abacus display of the core of B. Now the core of B is of the form $((p-z_2+i-1)^{z_1}, (i-1)^{z_2})$, with $i+z_2-1 \leq p$ and $z_2 \geq z_1$. The abacus display has three beads on the first $i-1$ runners, five beads on the next z_1 runners, four beads on the following $z_2 - z_1$ runners, and three beads on the remaining $p-i-z_2+1$ runners. See Fig. 7.

Restriction of irreducibles

Suppose μ is a p-regular partition of B. We can eliminate the possibility of D_k^μ occurring with multiplicity greater than 1 in any Specht module for most μ in B. If in the abacus display of μ there is a bead on runner $s \neq i, i+z_1, i+z_2, 1$ movable one space to the left onto runner $s-1$ and the sum of the weights of the beads on runner s is greater than the sum of the weights on runner $s-1$, it follows as in the rectangular core case that $D_k^\mu \downarrow_{B_s} = D_k^{\tilde{\mu}}$ is nonzero, where B_s is a block of defect 2. If μ has a bead on runner $s = 1$, $i+z_1$, or $i+z_2$ movable onto runner $s-1$, with no beads on runner $s-1$ at a higher position, then $D_k^\mu \downarrow_{B_s} = D_k^{\tilde{\mu}}$ is nonzero, where B_s is a block of defect 1 or 0. In these cases, the multiplicity of D_k^μ in any Specht module S_k^λ of B cannot be greater than 1 as all decomposition numbers of blocks of defect 0, 1, or 2 are either 0 or 1.

It follows from Lemma 3.1 that an irreducible D_k^μ can only occur with multiplicity greater than 1 in S_k^α, S_k^β, S_k^γ, and S_k^δ where α, β, γ, and δ are the usual partitions defined in a [3:2] pair. Therefore we need only consider irreducibles D_k^μ with $\mu > \delta$, $\mu \neq \alpha$, β, γ, or δ. (The multiplicities of D_k^α, D_k^β, D_k^γ, and D_k^δ in S_k^α, S_k^β, S_k^γ, and S_k^δ are dealt with in Lemma 3.4.) Now suppose μ has its bead of highest position at $5p+i-1$ on runner i. Then by Theorem 1.2, the multiplicities of D_k^μ in S_k^α, S_k^β, and S_k^γ are the same as the multiplicities of $D_k^{\mu^R}$ in $S_k^{\alpha^R}$, $S_k^{\beta^R}$, and $S_k^{\gamma^R}$, respectively, where μ^R, α^R, β^R, and γ^R are the partitions μ, α, β, and γ

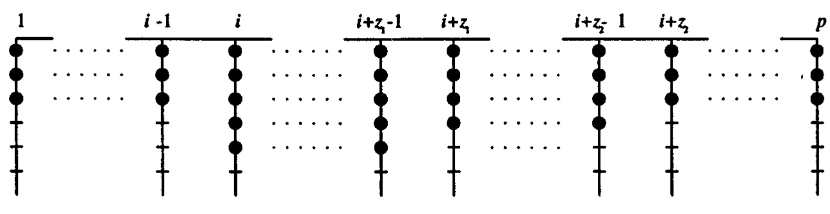


FIG. 7. Display of core of B.

with their first rows removed. Since μ^R is in a block of defect 2, the multiplicities are at most 1. Moreover, the partition α^R is always p-regular and $\mu^R \neq \alpha^R$, so $D_k^{\mu^R}$ can occur as a composition factor in at most two of the three Specht modules $S_k^{\alpha^R}$, $S_k^{\beta^R}$, and $S_k^{\gamma^R}$, because the nondiagonal entries of Cartan matrices of blocks of defect 2 are 0, 1, or 2 (see Scopes [11, 12]). Thus the multiplicity of D_k^{μ} is 1 in at most two of S_k^{α}, S_k^{β}, and S_k^{γ} and zero otherwise. By a similar argument, the multiplicity of $D_k^{\bar{\mu}}$ is 1 in at most two of $S_k^{\bar{\beta}}$, $S_k^{\bar{\gamma}}$, and $S_k^{\bar{\delta}}$ in $\bar{B}_i$ and zero otherwise. This information together with (A_2)–(D_2), implies that D_k^{μ} has multiplicity at most 1 in each of S_k^{α}, S_k^{β}, S_k^{γ}, and S_k^{δ}. The partitions unaccounted for are listed below in Table 4.

Sequences of blocks

As in the rectangular core case, we can reduce the list in Table 4 by considering restricting irreducibles through a sequence of blocks. First suppose $\mu \in B$ appears in Table 4 and has a bead of weight 2 on runner i at position $6p + i - 1$, and let μ_j be the partition given by moving this bead from position $6p + i - 1$ to $6p + i - j$, for $1 \leq j \leq p - z_1 + 1$. Then we have a sequence of partitions $\mu = \mu_1, \mu_2, \dots, \mu_{p-z_1+1} = \tilde{\mu}$ and blocks $B = \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_{p-z_1+1} = \tilde{B}$ such that $D_k^{\mu_j} \downarrow_{\mathfrak{B}_{j+1}}^{\mathfrak{B}_j}$ has a composition factor $D_k^{\mu_{j+1}}$. So by a series of restrictions of the irreducible D_k^{μ}, we obtain a nonzero module of $\tilde{B}$ with a composition factor $D_k^{\tilde{\mu}}$, where $\tilde{\mu}$ has a bead on runner $i + z_1$ at position $5p + i + z_1 - 1$. Suppose in addition that the display of $\mu \in B$ has five beads on runner $i + z_1 - 1$ of zero weight. Then the bead at position $5p + i + z_1 - 1$ in the display of $\tilde{\mu}$ can be moved one space to the left onto runner $i + z_1 - 1$ to give a partition ρ of a block $\mathfrak{B}$ such that $D_k^{\tilde{\mu}} \downarrow_{\mathfrak{B}}^{\tilde{B}} = D_k^{\rho}$. Note that the display of the core of $\mathfrak{B}$ has six beads on runner $i + z_1 - 1$. So we deduce that $D_k^{\mu} \downarrow_{\mathfrak{B}}^{B}$ is nonzero with a

TABLE 4

μ	Conditions on i, z_1, z_2
$\langle i \rangle$	—
$\langle i, i+1 \rangle$	$i \neq p$; $z_2 \geq 2$
$\langle i, i+z_1 \rangle$	$i + z_1 \leq p$
$\langle i, i+z_2 \rangle$	$z_2 > z_1$; $i + z_2 \leq p$
$\langle i, i \rangle$	—
$\langle i+2, i+1, i \rangle$	$z_1 \geq 3$; $i \leq p - 2$
$\langle i+z_1, i+1, i \rangle$	$z_1 \geq 2$; $i + z_1 \leq p$
$\langle i+z_2, i+1, i \rangle$	$z_1 \geq 2$; $i + z_2 \leq p$
$\langle i+1, i, i \rangle$	$z_1 \geq 2$; $i \neq p$
$\langle i+1, i \rangle$	$z_1 = 1$; $i \neq p$

composition factor D_k^{ρ}. The Specht modules S_k^{α}, S_k^{β}, S_k^{γ}, and S_k^{δ}, however, are zero on restriction to the block $\mathfrak{B}$, since it is impossible to obtain a partition of $\mathfrak{B}$ by moving beads in the displays of α, β, γ, and δ onto lower positions. Therefore, in this case D_k^{μ} does not occur as a composition factor in any S_k^{α}, S_k^{β}, S_k^{γ}, and S_k^{δ}.

It follows that we can eliminate the possibility of D_k^{μ} occurring as a composition factor of S_k^{α}, S_k^{β}, S_k^{γ}, and S_k^{δ} for the following cases: $\mu = \langle i, i + z_1 \rangle$, $\langle i, i \rangle$, and $\langle i, i + z_2 \rangle$, for $z_1 \geq 2$, and $\mu = \langle i, i+1 \rangle$, for $z_1 \geq 3$. Each of these satisfy the condition of having five beads of zero weight on runner $i + z_1 - 1$. The above argument is also valid for the case $\mu = \langle i \rangle$, for $z_1 \geq 2$.

Now suppose $\mu \in B$ appears in Table 4 and has a bead of weight 1 on runner i (at position $5p + i - 1$) and on runner $i + 1$ (at position $5p + i$), where $z_1 \geq 2$, and let $\overline{\mu}_j$ be the partition given by moving these beads onto positions $5p + i - j$ and $5p + i - j + 1$, respectively, for $1 \leq j \leq p - z_2 + 1$. Then we have a second sequence of partitions $\mu = \overline{\mu}_1, \overline{\mu}_2, \dots, \overline{\mu}_{p-z_2+1} = \breve{\mu}$, and blocks $B = \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_{p-z_2+1} = \breve{B}$ such that $D_k^{\overline{\mu}_j} \downarrow^{\mathfrak{B}_j}_{\mathfrak{B}_{j+1}}$ has a composition factor $D_k^{\overline{\mu}_{j+1}}$. So by a series of restrictions of the irreducible D_k^{μ}, we obtain a nonzero module of $\breve{B}$ with a composition factor $D_k^{\breve{\mu}}$, where $\breve{\mu}$ has beads at positions $4p + i + z_2 - 1$ and $4p + i + z_2$. Now assume in addition that the display of $\mu \in B$ has four beads on runners $i + z_2 - 2$ and $i + z_1 - 1$ all of zero weight. Then we can move the beads at positions $4p + i + z_2 - 1$ and $4p + i + z_2$ in the display of $\breve{\mu}$ two spaces to the left to give a partition ϵ of a block $\mathfrak{B}$ such that $D_k^{\breve{\mu}} \downarrow^{\breve{B}}_{\mathfrak{B}}$ has a composition factor D_k^{ϵ}. The display of the core of $\mathfrak{B}$ has five beads on runners $i + z_2 - 2$ and $i + z_2 - 1$. We deduce that $D_k^{\mu} \downarrow^{B}_{\mathfrak{B}}$ is nonzero with a composition factor D_k^{ϵ}. Again the Specht modules S_k^{α}, S_k^{β}, S_k^{γ}, and S_k^{δ} are zero on restriction to the block $\mathfrak{B}$, since the partitions α, β, γ, and δ do not have beads of weight 1 on runner $i + 1$. Therefore D_k^{μ} does not occur in S_k^{α}, S_k^{β}, S_k^{γ}, or S_k^{δ}.

We deduce that D_k^{μ} cannot occur in S_k^{α}, S_k^{β}, S_k^{γ}, or S_k^{δ} in the following cases: $\mu = \langle i+2, i+1, i \rangle$, $\langle i + z_2, i+1, i \rangle$, and $\langle i+1, i, i \rangle$, for $z_2 \geq z_1 + 2$ and $\mu = \langle i + z_1, i+1, i \rangle$, for $z_2 \geq z_1 + 3$. Each of these satisfy the condition of having four beads of zero weight on runners $i + z_2 - 1$ and $i + z_2 - 1$. We can also eliminate the case $\langle i+1, i \rangle$ where $z_1 = 1$, for values of $z_2 \geq 4$, by the same process.

We summarize conditions on z_1 and z_2 still to be accounted for in Table 5 below. For information on these remaining partitions, we look to the conjugate blocks. A list of μ^* is given in Table 6. As in the rectangular core case, we are able to eliminate a large number of μ where μ^* appears too far down the lexicographic ordering (these cases are labelled with †).

TABLE 5

μ	Conditions on z_1, z_2
$\langle i \rangle$	$z_1 = 1$
$\langle i, i+1 \rangle$	$z_1 = 2$
$\langle i, i+z_1 \rangle$	$z_1 = 1$
$\langle i, i+z_2 \rangle$	$z_1 = 1$
$\langle i, i \rangle$	$z_1 = 1$
$\langle i+2, i+1, i \rangle$	$(z_1 \geq 3)$ and $(z_2 = z_1$ or $z_2 = z_1 + 1)$
$\langle i+z_1, i+1, i \rangle$	$(z_1 \geq 2)$ and $(z_2 = z_1 + 1$ or $z_2 = z_1 + 2)$
$\langle i+z_2, i+1, i \rangle$	$(z_1 \geq 2)$ and $(z_2 = z_1$ or $z_2 = z_1 + 1)$
$\langle i+1, i, i \rangle$	$(z_1 \geq 2)$ and $(z_2 = z_1$ or $z_2 = z_1 + 1)$
$\langle i+1, i \rangle$	$(z_1 = 1)$ and $(z_2 = 2$ or $3)$

TABLE 6
Mullineux Mapping of Partitions of Table 5

$\mu = \langle i \rangle;\ z_1 = 1$	
Conditions on i, z_2	μ^*
$z_2 \geq 4$	$\langle p - z_2 + 4, p - z_2 + 3, p - z_2 + 2)$†
$z_2 = 3;\ i \leq p - 3$	$\langle p, p-1, i+1 \rangle$†
$z_2 = 3;\ i = p - 2$	$\langle p-1, p \rangle$†
$z_2 = 2;\ i \leq p - 3$	$\langle p, i+2, i+1 \rangle$†
$z_2 = 2;\ i = p - 2$	$\langle p, p-1 \rangle$†
$z_2 = 2;\ i = p - 1$	$\langle p, 2 \rangle$†
$z_2 = 1;\ i \leq p - 3$	$\langle i+3, i+2, i+1 \rangle$†
$z_2 = 1;\ i = p - 2$	$\langle p, p-1, 2 \rangle$†
$z_2 = 1;\ i = p - 1$	$\langle p, 3, 2 \rangle$
$z_2 = 1;\ i = p$	$\langle 4, 3, 2 \rangle$

$\mu = \langle i, i+1 \rangle;\ z_1 = 2$	
Conditions on i, z_2	μ^*
$z_2 \geq 5$	$\langle p - z_2 + 5, p - z_2 + 4, p - z_2 + 3 \rangle$†
$z_2 = 4;\ i \leq p - 4$	$\langle p, p-1, i+2 \rangle$†
$z_2 = 4;\ i = p - 3$	$\langle p-1, p \rangle$†
$z_2 = 3;\ i \leq p - 4$	$\langle p, i+3, i+2 \rangle$†
$z_2 = 3;\ i = p - 3$	$\langle p, p-1 \rangle$†
$z_2 = 3;\ i = p - 2$	$\langle p, 3 \rangle$†
$z_2 = 2;\ i \leq p - 4$	$\langle i+4, i+3, i+2 \rangle$†
$z_2 = 2;\ i = p - 3$	$\langle p, p-1, 3 \rangle$†
$z_2 = 2;\ i = p - 2$	$\langle p, 4, 3 \rangle$
$z_2 = 2;\ i = p - 1$	$\langle 5, 4, 3 \rangle$

TABLE 6—(*Continued*)

$\mu = \langle i+2, i+1, i\rangle$; $z_2 = z_1$ or $z_2 = z_1 + 1$	
Conditions on i, z_2	μ^*
$z_2 = z_1 + 1$; $i + z_2 \leq p - 1$	$\langle p, i + z_1 + 1, i + z_1\rangle$†
$z_2 = z_1 + 1$; $i + z_2 = p$	$\langle p, p - 1\rangle$†
$z_2 = z_1 + 1$; $i + z_2 = p + 1$	$\langle p, z_1 + 1\rangle$†
$z_2 = z_1 \leq p - 3$; $i + z_2 \leq p - 2$	$\langle i + z_1 + 2, i + z_1 + 1, i + z_1\rangle$†
$z_2 = z_1 \leq p - 3$; $i + z_2 = p - 1$	$\langle p, p - 1, z_1 + 1\rangle$†
$z_2 = z_1 \leq p - 3$; $i + z_2 = p$	$\langle p, z_1 + 2, z_1 + 1\rangle$
$z_2 = z_1 \leq p - 3$; $i + z_2 = p + 1$	$\langle z_1 + 3, z_1 + 2, z_1 + 1\rangle$
$z_2 = z_1 = p - 2$; $i = 2$	$\langle p, p - 1\rangle$
$z_2 = z_1 = p - 2$; $i = 3$	$\langle p - 1, p\rangle$
$z_2 = z_1 = p - 1$; $i = 2$	$\langle p\rangle$
$\mu = \langle i+z, i+1, i\rangle$; $z_2 = z_1 + 2$ or $z_2 = z_1 + 1$	
Conditions on i, z_2	μ^*
$z_2 = z_1 + 2$; $i + z_2 \leq p - 1$	$\langle p - 1, i + z_1 + 1, i + z_1\rangle$†
$z_2 = z_1 + 2$; $i + z_2 = p$	$\langle p - 1, p - 2\rangle$†
$z_2 = z_1 + 2$; $i + z_2 = p + 1$	$\langle p - 1, z_1 + 1\rangle$†
$z_1 + 1 = z_2 \leq p - 3$; $i + z_2 \leq p - 2$	$\langle i + z_1 + 2, i + z_1 + 1, i + z_1\rangle$†
$z_1 + 1 = z_2 \leq p - 3$; $i + z_2 = p - 1$	$\langle p - 1, p - 2, z_1 + 1\rangle$†
$z_1 + 1 = z_2 \leq p - 3$; $i + z_2 = p$	$\langle p - 1, z_1 + 2, z_1 + 1\rangle$
$z_1 + 1 = z_2 \leq p - 3$; $i + z_2 = p + 1$	$\langle z_1 + 3, z_1 + 2, z_1 + 1\rangle$
$z_1 + 1 = z_2 = p - 2$; $i = 2$	$\langle p - 1, p - 2\rangle$
$z_1 + 1 = z_2 = p - 2$; $i = 3$	$\langle p - 2, p - 1\rangle$
$z_1 + 1 = z_2 = p - 1$; $i = 2$	$\langle p - 1\rangle$
$\mu = \langle i, i + z_1\rangle$; $z_1 = 1$	
Conditions on i, z_2	μ^*
$z_2 > 1$; $i + z_2 \leq p$	$\langle p - z_2 + 2, i + 1, 1\rangle$†
$z_2 > 1$; $i + z_2 = p + 1$	$\langle p - z_2 + 2, 1\rangle$†
$z_2 = 1$	$\langle i + 1, 2, 2\rangle$†
$\mu = \langle i, i + z_2\rangle$; $z_1 = 1$	
Conditions on i, z_2	μ^*
$z_2 \geq 3$	$\langle p - z_2 + 3, p - z_2 + 2, 2\rangle$†
$z_2 = 2$	$\langle p, 2, 2\rangle$†

TABLE 6—(*Continued*)

$\mu = \langle i, i\rangle;\ z_1 = 1$	
Conditions on i, z_2	μ^*
$z_2 = 1;\ i \le p-2$	$\langle i+2, i+1, i+1\rangle\dagger$
$z_2 = 1;\ i = p-1$	$\langle p, p, 2\rangle\dagger$
$z_2 = 1;\ i = p$	$\langle 3, 2, 2\rangle$
$z_2 \ge 2;\ i + z_2 \le p-1$	$\langle p - z_2 + 2, i+1, i+1\rangle\dagger$
$z_2 \ge 2;\ i + z_2 = p$	$\langle i+2, i+1, i+1\rangle\dagger$
$z_2 \ge 3;\ i + z_2 = p+1$	$\langle p - z_2 + 3, p - z_2 + 2, 2\rangle\dagger$
$z_2 = 2;\ i = p-1$	$\langle p, 2, 2\rangle\dagger$
$\mu = \langle i + z_2, i+1, i\rangle;\ z_2 = z_1$ or $z_2 = z_1 + 1$	
Conditions on i, z_2	μ^*
$z_2 = z_1 + 1$	$\langle p, z_1+1, z_1+1\rangle\dagger$
$z_2 = z_1$	$\langle i + z_1, z_1+1, z_1+1\rangle\dagger$
$\mu = \langle i+1, i, i\rangle;\ z_2 = z_1$ or $z_2 = z_1 + 1$	
Conditions on i, z_2	μ^*
$z_2 = z_1 + 1;\ i + z_2 \le p$	$\langle p, i + z_1, i + z_1\rangle\dagger$
$z_2 = z_1 + 1;\ i + z_2 = p+1$	$\langle p, z_1+1, z_1+1\rangle\dagger$
$z_2 = z_1;\ i + z_2 \le p-1$	$\langle i + z_1 + 1, i + z_1, i + z_1\rangle\dagger$
$z_2 = z_1;\ i + z_2 = p$	$\langle p, p, z_1+1\rangle\dagger$
$z_2 = z_1 \le p-2;\ i + z_2 = p+1$	$\langle z_1+2, z_1+1, z_1+1\rangle$
$z_2 = z_1 = p-1;\ i = 2$	$\langle p, p\rangle$
$\mu = \langle i+1, i\rangle;\ z_2 = 2$ or 3	
Conditions on i, z_2	μ^*
$z_2 = 2;\ i \le p-4$	$\langle i+3, i+2, i+1\rangle\dagger$
$z_2 = 2;\ i = p-3$	$\langle p-1, p-2, 2\rangle\dagger$
$z_2 = 2;\ i = p-2$	$\langle p-1, 3, 2\rangle$
$z_2 = 2;\ i = p-1$	$\langle 4, 3, 2\rangle$
$z_2 = 3;\ i \le p-4$	$\langle p-1, i+2, i+1\rangle\dagger$
$z_2 = 3;\ i = p-3$	$\langle p-1, p-2\rangle\dagger$
$z_2 = 3;\ i = p-2$	$\langle p-1, 2\rangle\dagger$

6. EXTENSIONS

6.1. *The rectangular core case*

Suppose the block B of $\mathfrak{S}_n$ has rectangular core $(i-1)^z$. Recall from the proof of Lemma 2.4 that there is a block $\hat{B}$ of defect 1 of $\mathfrak{S}_{n+1}$ such that the display of its core has one more bead on runner i and one less bead on runner $i-1$ than that of the core of B. Then there is a permutation π of $\{1,\dots,p\}$ such that ${}^{i}\alpha_{\pi(1)} > {}^{i}\alpha_{\pi(2)} > \cdots > {}^{i}\alpha_{\pi(p)}$ in B and $\langle \pi(1)\rangle > \langle \pi(2)\rangle > \cdots > \langle \pi(p)\rangle$ in $\hat{B}$. In particular,

$$S_k^{\langle \pi(c)\rangle} \downarrow_B^{\hat{B}} \sim \begin{array}{c} S_k^{{}^{i}\alpha_{\pi(c)}} \\ S_k^{{}^{i}\beta_{\pi(c)}} \\ S_k^{{}^{i}\gamma_{\pi(c)}} \end{array} .$$

By Frobenius Reciprocity, this implies that multiple or self-extensions of $D_k^{{}^{i}\alpha^*_{\pi(c)}}$ can only occur when at least one of ${}^{i}\beta_{\pi(c)}$ and ${}^{i}\gamma_{\pi(c)}$ is p-singular. Moreover, multiple or self-extensions of $D_k^{{}^{i}\alpha_{\pi(c)}}$ exist if and only if they exist for $D_k^{{}^{i}\alpha^*_{\pi(c)}}$ in the conjugate block B' with core z^{i-1}. If ${}^{i}\alpha^*_{\pi(c)} = {}^{z+1}\alpha_h$ in B', for some $1 \le h \le p$, then multiple or self-extensions of $D_k^{{}^{i}\alpha_{\pi(c)}}$ exist only when at least one of ${}^{z+1}\beta_h$ and ${}^{z+1}\gamma_h$ is p-singular. Therefore we need only consider cases of $D_k^{{}^{i}\alpha_{\pi(c)}}$ when at least one of ${}^{i}\beta_{\pi(c)}$ and ${}^{i}\gamma_{\pi(c)}$ and at least one of ${}^{z+1}\beta_h$ and ${}^{z+1}\gamma_h$ are p-singular. By appealing to conjugacy in this way, the only case left to consider is the conjugate pair $D_k^{{}^{2}\alpha_2}$ and $D_k^{{}^{3}\alpha_1}$ of irreducibles in the blocks of $\mathfrak{S}_{3p+2}$ with cores (1^2) and (2), respectively. The matrix of composition factors given below of $S_k^{{}^{2}\alpha_2}$, $S_k^{{}^{2}\beta_2}$, and $S_k^{{}^{2}\gamma_2}$ in the block B of $\mathfrak{S}_{3p+2}$ with core (1^2) can be calculated using Schaper's formula:

$$\begin{array}{cccccc}
 & D_k^{\langle 3,4\rangle} & D_k^{\langle 3,3\rangle} & D_k^{\langle 2,4\rangle} & D_k^{\langle 2,2\rangle} & D_k^{\langle 1\rangle} \\
S_k^{\langle 2,2\rangle} & 1 & 1 & 1 & 1 & \\
S_k^{\langle 2,1\rangle} & & 1 & & 1 & \\
S_k^{\langle 1\rangle} & & & 1 & 1 & 1
\end{array} .$$

Since $S_k^{\langle 2\rangle} = D_k^{\langle 2\rangle}$ in the block $\hat{B}$ of $\mathfrak{S}_{3p+3}$, the indecomposable $S_k^{\langle 2\rangle} \downarrow_B^{\hat{B}}$ is self-dual. By obtaining information about the Ext-quiver of B from the Ext-quivers of the principal blocks of $\mathfrak{S}_{3p}$ and $\mathfrak{S}_{3p+1}$, we can show that $S_k^{\langle 2\rangle} \downarrow_B^{\hat{B}}$ has Loewy series

$$\begin{array}{c}
D_k^{\langle 2,2\rangle} \\
D_k^{\langle 3,3\rangle} D_k^{\langle 2,4\rangle} \\
D_k^{\langle 3,4\rangle} D_k^{\langle 2,2\rangle} D_k^{\langle 1\rangle} \\
D_k^{\langle 3,3\rangle} D_k^{\langle 2,4\rangle} \\
D_k^{\langle 2,2\rangle}
\end{array} .$$

Therefore multiple or self-extensions do not occur in this remaining case.

6.2. *The birectangular core case*

Suppose the blocks B and $\overline{B}$ form a [3:2] pair and that $P(D_k^\lambda)$ is a typical projective indecomposable module in B which is not isomorphic to $P(D_k^\alpha)$. Assuming decomposition numbers are all 0 or 1, $P(D_k^\lambda)$ has at most four copies of D_k^α. Then by inducing and restricting projective indecomposables, we may show that:

(1) $P(D_k^\lambda)$ has no copies of D_k^α if and only if $P(D_k^{\overline{\lambda}})$ has no copies of $D_k^{\overline{\alpha}}$,

(2) $P(D_k^\lambda)$ has one copy of D_k^α if and only if $P(D_k^{\overline{\lambda}})$ has three copies of $D_k^{\overline{\alpha}}$,

(3) $P(D_k^\lambda)$ has two copies of D_k^α if and only if $P(D_k^{\overline{\lambda}})$ has two copies of $D_k^{\overline{\alpha}}$,

(4) $P(D_k^\lambda)$ has three copies of D_k^α if and only if $P(D_k^{\overline{\lambda}})$ has one copy of $D_k^{\overline{\alpha}}$,

(5) $P(D_k^\lambda)$ has four copies of D_k^α if and only if $P(D_k^{\overline{\lambda}})$ has four copies of $D_k^{\overline{\alpha}}$.

(Here λ and $\overline{\lambda}$ are such that $D_k^\lambda\downarrow_{\overline{B}} \cong D_k^{\overline{\lambda}} \oplus D_k^{\overline{\lambda}}$ and $D_k^{\overline{\lambda}}\uparrow^B \cong D_k^\lambda \oplus D_k^\lambda$. The notation has to be modified slightly when $\lambda = \beta$, γ, or δ.)

For example, if $P(D_k^\lambda)$ has one copy of D_k^α, the $P(D_k^\lambda)\downarrow_{\overline{B}}\uparrow^B$ has thirty-six copies of D_k^α and is isomorphic to the direct sum of four copies of $P(D_k^\lambda)$ and eight copies of $P(D_k^\alpha)$. So $P(D_k^\lambda)\downarrow_{\overline{B}}$ is isomorphic to $P(D_k^{\overline{\lambda}}) \oplus P(D_k^{\overline{\lambda}})$ and $P(D_k^{\overline{\lambda}})\uparrow^B$ is isomorphic to the direct sum of two copies of $P(D_k^\lambda)$ and four copies of $P(D_k^\alpha)$. Hence $P(D_k^{\overline{\lambda}})$ has three copies of $D_k^{\overline{\alpha}}$.

Furthermore, if there exists a projective $P(D_k^{\overline{\lambda}})$ in $\overline{B}$ with only one copy of $D_k^{\overline{\alpha}}$, then since $P(D_k^{\overline{\lambda}})\uparrow^B$ is isomorphic to $P(D_k^\lambda) \oplus P(D_k^\lambda)$ in this case, it follows that $D_k^{\overline{\alpha}}\uparrow^B$ is the direct sum of precisely two identical indecomposable modules. Each indecomposable has head and socle equal to D_k^α and takes one of the following two forms:

$$\begin{array}{ccc} D_k^\alpha & & D_k^\alpha \\ D_k^\alpha \oplus N_1 & ; & N_2 \\ D_k^\alpha & & D_k^\alpha \end{array}$$

where N_1 is a self-dual module with no copies of D_k^α and N_2 is self-dual whose single copy of D_k^α does not appear in its head and socle. Both N_1 and N_2 have two copies of D_k^λ.

Now suppose the block B of $\mathfrak{S}_n$ has birectangular core $((p - z_2 + i - 1)^{z_1}, (i-1)^{z_2})$, with $i + z_2 - 1 \le p$ and $z_2 \ge z_1$. Then multiple and self-extensions of $D_k^{{}^i\alpha}$ can only exist when at least one of ${}^i\beta$, ${}^i\gamma$, or ${}^i\delta$ is

p-singular. This only occurs when $i = 2$. By appealing to the conjugate block B' with core $((z_1 + z_2)^{i-1}, z_1^{p-z_2})$, we also require $z_1 = 1$. So we need only consider blocks with cores of the form $(p - z_2 + 1, 1^{z_2})$. In such a block, ${}^2\beta$ is always p-singular, ${}^2\delta$ is always p-regular, and ${}^2\gamma$ is p-singular if $z_2 = p - 1$ and p-regular otherwise.

We first consider self-extensions in a defect 3 block B with core $(p - z_2 + 1, 1^{z_2})$, where $2 \le z_2 \le p - 2$. Since any Specht module S_k^{λ} is isomorphic to the dual of $S_k^{\lambda'} \otimes alt$, it follows that both $S_k^{{}^2\alpha}$ and $S_k^{{}^2\delta}$ in B have irreducible heads and socles, $S_k^{{}^2\beta}$ has an irreducible socle, and $S_k^{{}^2\gamma}$ has an irreducible head. By Corollary 3.6, self-extensions of $D_k^{{}^2\alpha}$ can only exist if $D_k^{{}^2\alpha}$ appears in the head of $S_k^{{}^2\beta}$. By appealing to conjugacy, this occurs if and only if $D_k^{{}^2\alpha}$ appears in the socle of $S_k^{{}^2\gamma}$. Therefore, if $D_k^{{}^2\alpha}$ extends itself, there must be a copy of $D_k^{{}^2\alpha}$ in the head, socle, second Loewy layer, and second socle layer of $P(D_k^{{}^2\alpha})$. This is impossible, however, by consideration of the possible structures of $D_k^{{}^2\overline{\alpha}} \uparrow^B$ above. (Note that $P(D_k^{\langle z_2+2,1,1\rangle})$ in $\overline{B}_2$ has exactly one copy of $D_k^{{}^2\overline{\alpha}}$, so $D_k^{{}^2\overline{\alpha}} \uparrow^B$ does indeed take one of the two forms.)

Now suppose multiple extensions occur in such a block. Then by Corollary 3.6, there must exist an irreducible D_k^{λ} in both the second Loewy layer of $S_k^{{}^2\alpha}$ and the head of $S_k^{{}^2\beta}$. By conjugacy, D_k^{λ} also appears in the socle of $S_k^{{}^2\gamma}$ and in the second socle layer of $S_k^{{}^2\delta}$. So there is an irreducible D_k^{λ} with multiplicity 1 in $S_k^{{}^2\alpha}$, $S_k^{{}^2\beta}$, $S_k^{{}^2\gamma}$, and $S_k^{{}^2\delta}$. We can eliminate this possibility, however, by the arguments of Section 5.2 and, in particular, by referring to Table 6.

It just remains to consider the pair of conjugate blocks of $\mathfrak{S}_{4p+1}$ with cores $(p, 1)$ and $(2, 1^{p-1})$. By Schaper's formula, the Specht module $S_k^{{}^2\beta}$ in the block B with core $(p, 1)$ is irreducible with a single factor $D_k^{{}^2\alpha}$, so multiple extensions cannot occur. If $D_k^{{}^2\alpha}$ in B extends itself, then either $D_k^{{}^2\alpha}$ appears in the head, socle, second Loewy layer, and second socle layer of $P(D_k^{{}^2\alpha})$ as above, or since $S_k^{{}^2\beta}$ is irreducible, $P(D_k^{{}^2\alpha})$ could be of the form

$$\begin{array}{c} D_k^{{}^2\alpha} \\ D_k^{{}^2\alpha} \oplus N \\ D_k^{{}^2\alpha} \end{array},$$

where N is a self-dual module whose single copy of $D_k^{{}^2\alpha}$ does not appear in its head and socle. By considering the possible structures of $D_k^{{}^2\overline{\alpha}} \uparrow^B$ again, both cases are impossible.

7. THE MAIN THEOREM

We conclude with a statement of the main theorem.

THEOREM 7.1. *Let $p \geq 5$. A block B of $k\mathfrak{S}_n$ of defect 3 has the following properties*:

(1) *All the decomposition numbers are* 0 *or* 1.

(2) $(D_k^\lambda, D_k^\lambda)^1_{k\mathfrak{S}_n} = 0$ *for all p-regular* λ.

(3) $(D_k^\lambda, D_k^\mu)^1_{k\mathfrak{S}_n} = 0$ *or* 1 *for all p-regular* λ *and* μ.

Remark. These properties are common to blocks of defect 2 and 3. The first property, however, does not hold for blocks of defect 4.

APPENDIX

We list in Tables 7 and 8 all the remaining cases where D_k^μ may occur with multiplicity greater than 1 in some Specht module.

These cases are all dealt with similarly by using Schaper's formula. We consider the first rectangular case $\mu = \langle p-1 \rangle$ in the block B with core $(p-2)$. So $\mu^* = \langle p, 3, 2 \rangle$ in the block B' with core $1^{(p-2)}$. (Further examples appear in [9].)

Now ${}^2\alpha_j$, ${}^2\beta_j$, and ${}^2\gamma_j$, when $j \in \{2, p-1, p-2, \ldots, 5, 4\}$, are all lexicographically greater than $\mu^* = \langle p, 3, 2 \rangle$ in the block B'. So the multiplicity of $D_k^{\langle p,3,2 \rangle}$ in any of $S_k^{{}^2\alpha_j}$, $S_k^{{}^2\beta_j}$, and $S_k^{{}^2\gamma_j}$ for those values of j is zero. Also by Theorem 1.2 (first row removal), the multiplicity of $D_k^{\langle p,3,2 \rangle}$ in $S_k^{{}^2\alpha_3}$,

TABLE 7
Rectangular Core Cases

μ	Core	μ^*	Core
$\langle p-1 \rangle$	$(p-2)$	$\langle p, 3, 2 \rangle$	1^{p-2}
$\langle p \rangle$	$(p-1)$	$\langle 4, 3, 2 \rangle$	1^{p-1}
$\langle p-2, p-1 \rangle$	$(p-3)^2$	$\langle p, 4, 3 \rangle$	2^{p-3}
$\langle p-1, p \rangle$	$(p-2)^2$	$\langle 5, 4, 3 \rangle$	2^{p-2}
$\langle i+2, i+1, i \rangle$	$(i-1)^z$; $i+z=p$	$\langle p, z+2, z+1 \rangle$	z^{i-1}
$\langle i+2, i+1, i \rangle$	$(i-1)^z$; $i+z=p+1$	$\langle z+3, z+2, z+1 \rangle$	z^{i-1}
$\langle p, p-1 \rangle$	$(p-2)$	$\langle 4, 3, 2 \rangle$	1^{p-2}

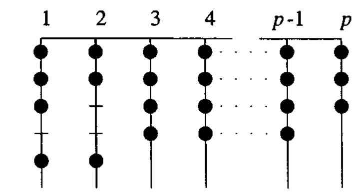
TABLE 8
Birectangular Core Cases

μ	Core	μ^*	Core
$\langle p-1\rangle$	$(2p-3, p-2)$	$\langle p, 3, 2\rangle$	$(2^{p-2}, 1^{p-1})$
$\langle p\rangle$	$(2p-2, p-1)$	$\langle 4, 3, 2\rangle$	$(2^{p-1}, 1^{p-1})$
$\langle p-2, p-1\rangle$	$((2p-5)^2, (p-3)^2)$	$\langle p, 4, 3\rangle$	$(4^{p-3}, 2^{p-2})$
$\langle p-1, p\rangle$	$((2p-4)^2, (p-2)^2)$	$\langle 5, 4, 3\rangle$	$(4^{p-2}, 2^{p-2})$
$\langle p, p\rangle$	$(2p-2, p-1)$	$\langle 3, 2, 2\rangle$	$(2^{p-1}, 1^{p-1})$
$\langle 4, 3, 2\rangle$	$(3^{p-2}, 1^{p-2})$	$\langle p, p-1\rangle$	$(2p-5, (p-2)^2)$
$\langle p-1, 3, 2\rangle$	$(3^{p-3}, 1^{p-2})$	$\langle p-1, p-2\rangle$	$(2p-5, (p-3)^2)$
$\langle i+2, i+1, i\rangle$		$\langle p, z_1+2, z_1+1\rangle$	
$\langle i+2, i+1, i\rangle$	See	$\langle z_1+3, z_1+2, z_1+1\rangle$	See
$\langle p-1, i+1, i\rangle$	Table 6	$\langle p-1, z_1+2, z_1+1\rangle$	Table 6
$\langle i+1, i, i\rangle$		$\langle z_1+2, z_1+1, z_1+1\rangle$	

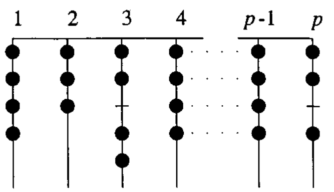
$S_k^{2\beta_3}$, and $S_k^{2\gamma_3}$ is at most 1. So it just remains to consider the multiplicity of $D_k^{\langle p,3,2\rangle}$ in $S_k^{2\alpha_p}$, $S_k^{2\beta_p}$, and $S_k^{2\gamma_p}$, and $S_k^{2\alpha_1}$, $S_k^{2\beta_1}$, $S_k^{2\gamma_1}$.

We first consider the multiplicity of $D_k^{\langle p,3,2\rangle}$ in $S_k^{2\alpha_p}$, $S_k^{2\beta_p}$, and $S_k^{2\gamma_p}$. To do this, we consider the multiplicity of $D_k^{\langle p,3,1\rangle} = D_k^{\langle p,3,2\rangle}\downarrow_{\overline{B}'}$ in $S^{2\overline{\alpha}_{pk}}$, $S_k^{2\overline{\beta}_p}$, and $S_k^{2\overline{\gamma}_p}$ (which is assumed to be 0 or 1 in each case) and then refer to statements (1)–(4) of Section 5.1. We use the notation of [7, Section 1.2] (letting $\nu = \nu_p(x - m_c) - \nu_p(y - m_c)$).

First note that the multiplicity of $D_k^{\langle p,3,1\rangle}$ in $S_k^{\langle 3,3,1\rangle}$, $S_k^{\langle 3,2,1\rangle}$, and $S_k^{\langle 3,1,1\rangle}$ is 1, 0, and 0, respectively by Theorem 1.2 (row reduction). For each λ below, we list every partition μ with $\langle p,3,1\rangle \geq \mu > \lambda$ that occurs as a contribution of S_k^{λ} by Schaper's formula (M denotes the multiplicity of $D_k^{\langle p,3,1\rangle}$ in S_k^{μ}). We are therefore able to calculate the multiplicity of $D_k^{\langle p,3,1\rangle}$ in S_k^{λ}.

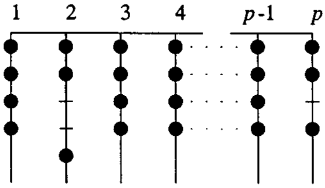


x	y	m_c	ν	$\#_{x,y,c}$	μ	M
$4p+1$	$2p+2$	$2p+1$	1	0	$\langle 3,3,1\rangle$	1
$4p+1$	$3p+2$	$3p+1$	1	0	$\langle 3,2,1\rangle$	0

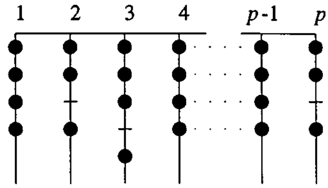


x	y	m_c	ν	$\#_{x,y,c}$	μ	M
$3p+2$	$3p$	$2p+2$	1	$2p-7$	$\langle p,3,1\rangle$	1
$4p-1$	$3p$	$3p-1$	1	0	$\langle 3,3,1\rangle$	1

Hence the multiplicity of $D_k^{\langle p,3,1\rangle}$ in $S_k^{\langle 2,1\rangle}$ and $S_k^{\langle p,3,3\rangle}$ is 1 and 0, respectively.

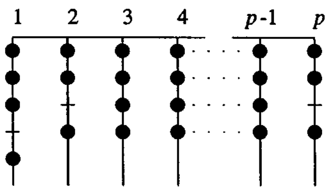


x	y	m_c	ν	$\#_{x,y,c}$	μ	M
$4p+1$	$2p+2$	$2p+1$	1	0	$\langle p,3,3\rangle$	0
$4p-1$	$3p$	$3p-1$	1	0	$\langle 2,1\rangle$	1
$4p+1$	$3p+2$	$3p+1$	1	0	$\langle p,3,2\rangle$	1

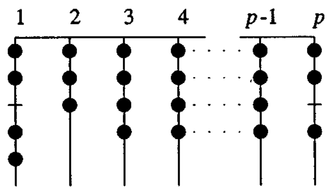


x	y	m_c	ν	$\#_{x,y,c}$	μ	M
$3p+1$	$2p+2$	$2p+1$	1	0	$\langle p,3,3\rangle$	0
$3p+1$	$3p$	$2p+1$	1	$2p-6$	$\langle p,3,1\rangle$	1
$4p-1$	$3p$	$3p-1$	1	0	$\langle 3,2,1\rangle$	0

This implies that $D_k^{\langle p,3,1\rangle}$ occurs with multiplicity 1 in both $S_k^{{}^2\overline{\alpha}_p}$ and $S_k^{\langle p,3,2\rangle}$.



x	y	m_c	ν	$\#_{x,y,c}$	μ	M
$3p+2$	$3p+1$	$2p+1$	-1	$2p-5$	$\langle p,3,1\rangle$	1
$4p-1$	$3p+1$	$3p-1$	1	1	$\langle 2,1\rangle$	1
$4p-1$	$3p+2$	$3p-1$	1	2	$\langle 3,2,1\rangle$	0
$4p$	$3p+1$	$3p$	1	0	$\langle 2,p\rangle$	1
$4p$	$3p+2$	$3p$	1	1	$\langle p,3,2\rangle$	1

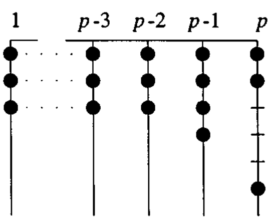


x	y	m_c	ν	$\#_{x,y,c}$	μ	M
$3p$	$2p+1$	$2p$	1	0	$\langle p,2,1\rangle$	0
$4p$	$2p+1$	$2p$	1	0	$\langle 2,p\rangle$	1
$4p$	$2p+2$	$2p$	1	1	$\langle p,3,3\rangle$	0
$3p+2$	$3p$	$2p$	-1	$2p-4$	$\langle p,3,1\rangle$	1
$4p-1$	$3p+2$	$3p-1$	1	2	$\langle 3,1,1\rangle$	0

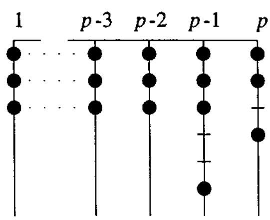
So $D_k^{\langle p,3,1\rangle}$ occurs with multiplicity 0 in both $S_k^{{}^2\overline{\beta}_p}$ and $S_k^{{}^2\overline{\gamma}_p}$.

Therefore $D_k^{\langle p,3,2\rangle}$ cannot occur with multiplicity 2 in $S_k^{{}^2\alpha_p}$, $S_k^{{}^2\beta_p}$, or $S_k^{{}^2\gamma_p}$ in B' since by statements (1)–(4) of Section 5.1, this can only happen when $D_k^{\langle p,3,1\rangle}$ occurs with multiplicity 1 in at least two of $S_k^{{}^2\overline{\alpha}_p}$, $S_k^{{}^2\overline{\beta}_p}$, and $S_k^{{}^2\overline{\gamma}_p}$ in $\overline{B}'$.

Under conjugation, the partitions ${}^{p-1}\alpha_{p-1}$, ${}^{p-1}\beta_{p-1}$, and ${}^{p-1}\gamma_{p-1}$ in B map to ${}^2\gamma_1$, ${}^2\beta_1$, and ${}^2\alpha_1$, respectively, in B'. Therefore, it is enough to consider the multiplicity of $D_k^{\langle p-1\rangle}$ in $S_k^{{}^{p-1}\alpha_{p-1}}$, $S_k^{{}^{p-1}\beta_{p-1}}$, and $S_k^{{}^{p-1}\gamma_{p-1}}$.

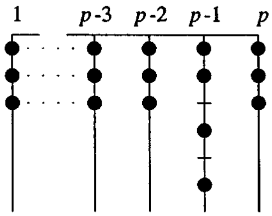


x	y	m_c	ν	$\#_{x,y,c}$	μ	M
$6p-1$	$4p-2$	$3p-1$	1	0	$\langle p-1\rangle$	1

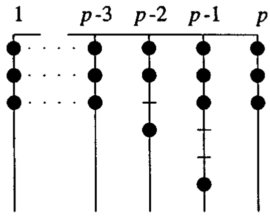


x	y	m_c	ν	$\#_{x,y,c}$	μ	M
$6p-2$	$4p-1$	$3p-1$	-1	0	$\langle p-1\rangle$	1
$6p-2$	$4p-1$	$4p-2$	1	0	$\langle p\rangle$	1

Hence the multiplicity of $D_k^{\langle p-1\rangle}$ in $S_k^{\langle p\rangle}$ and $S_k^{\langle p-1,p\rangle}$ is 1 and 0, respectively.



x	y	m_c	ν	$\#_{x,y,c}$	μ	M
$4p-2$	$3p-1$	$3p-2$	1	0	$\langle p-1,p\rangle$	0
$6p-2$	$3p-1$	$3p-2$	1	0	$\langle p\rangle$	1



x	y	m_c	ν	$\#_{x,y,c}$	μ	M
$4p-3$	$3p-2$	$3p-3$	1	0	$\langle p-1,p-1\rangle$	1
$4p-3$	$3p-1$	$3p-3$	1	1	$\langle p-1,p\rangle$	0
$6p-2$	$4p-3$	$3p-3$	-1	2	$\langle p-1\rangle$	1

So the multiplicity of $D_k^{\langle p-1\rangle}$ in $S_k^{{}^1\alpha_{p-1}}$ and $S_k^{{}^{p-1}\beta_{p-1}}$ is 1 and 0, respectively. So by statements (1)–(4) of Section 5.1, the multiplicity of $D_k^{\langle p-1\rangle}$ in $S_k^{{}^{p-1}\gamma_{p-1}}$ is at most 1. Therefore $D_k^{\langle p-1\rangle}$ does not occur in any Specht module of B with multiplicity greater than 1. Equivalently, $D_k^{\langle p,3,2\rangle}$ does not occur in any Specht module of B' with multiplicity greater than 1.

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REFERENCES

1. D. J. Benson, "Representations and Cohomology," Vol. I, Cambridge Studies in Advanced Mathematics **30**, Cambridge University Press, London, 1991.
2. G. D. James, On the decomposition matrices of the symmetric groups III, *J. Algebra* **71** (1981), 115–122.
3. G. D. James and A. Kerber, The representation theory of the symmetric group, "Enclopedia of Mathematics," Vol. 16, Addison-Wesley, Reading, MA, 1981.
4. G. D. James and A. Mathas, A q-analogue of the Jantzen–Schaper theorem, *Proc. London Math. Soc.* (3) **74** (1997), 241–274.
5. P. Landrock, "Finite Group Algebras and their Modules," Lecture Notes in Mathematical Sciences **84**, Cambridge University Press, London, 1983.
6. S. Martin, On the ordinary quiver of the principal block of certain symmetric groups, *Quart. J. Math. Oxford* (2) **40** (1989), 209–223.
7. S. Martin and L. Russell, On the Ext-quiver of blocks of defect 3 of symmetric group algebras, *J. Algebra* **185** (1996), 440–480.
8. G. Mullineux, Bijections of p-regular partitions and p-module irreducibles of the symmetric groups, *J. London Math. Soc.* (20 **20** (1979), 60–66.

9. L. Russell, Modular representations of the symmetric groups, Ph.D. thesis, Univ. of Cambridge, 1996.
10. K. D. Schaper, Charakterformeln für Weyl-Moduln und Specht-Moduln in Primcharakteristik, Diplomarbeit, Bonn, 1981.
11. J. C. Scopes, Cartan matrices and Morita equivalence for blocks of the symmetric groups, *J. Algebra* **142** (1991), 441–455.
12. J. C. Scopes, Symmetric group blocks of defect two, *Quart. J. Math. Oxford* (2) **46** (1995), 201–234.
13. K. M. Tan, On [3, 1] pairs, submitted.